POWER MINIMIZATION IN THE MULTIUSER DOWNLINK UNDER USER RATE CONSTRAINTS AND IMPERFECT TRANSMITTER CSI

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ABSTRACT

The aim of this work is to jointly achieve individual rate requirements and minimum total transmit power in the vector *Broadcast Channel* (BC). Data streams are transmitted from a multi-antenna base station to several non-cooperative single-antenna receivers having perfect *Channel-State-Information* (CSI). Partial CSI, e.g., obtained via feedback, is used for the design of linear transmit filters at the transmitter. Employing a duality between *Multiple Access Channel* (MAC) and BC rate regions and the so-called *standard* interference functions, we propose an algorithmic joint solution for the transmit filter design and the power allocation in this work.

1. INTRODUCTION

We consider the design of linear precoders in the vector BC assuming erroneous CSI at the transmitter but perfect CSI at the receivers. Based on the appropriate duality between the MAC and the BC, the BC problem can be reformulated in the dual MAC. Due to the assumption of erroneous CSI, however, the dualities presented in [1–6] cannot be applied. Instead, we have to resort to the duality shown in [7] allowing for a different level of CSI at the transmitter and the receivers. Additionally, we note that we do not apply the standard assumption that the CSI errors at the transmitter and the receivers are identical as done in [8] (see also the references given in [7]).

In [9–12], the precoder design was based on a model with bounded errors that is well suited for systems with feedback. For a stochastic error model, the average sum *Mean Square Error* (MSE) was minimized in [7, 8]. The precoder design under probabilistic constraints was considered in [13–15].

We employ a stochastic error model, e.g., resulting from estimation in the reverse link or feedback, and a formulation based on ergodic rates as in [16] where bounds to the achievable rates for linear zero-forcing precoders based on imperfect CSI were presented. However, as the optimization of a non-zero-forcing linear precoder based on the ergodic rates is difficult for partial CSI, we concentrate on lower bounds to the ergodic rates depending on the average MSE (see Section 3).

The minimization of the total transmit power under average per-user MSE constraints is considered. For perfect transmitter CSI, the joint power allocation and transceiver optimization for the MMSE balancing problem was solved in [17] by means of a standard interference function [18, 19]. However, assuming perfect transmitter CSI is unrealistic.

Our contribution is an algorithmic solution of the power minimization problem under the assumption of imperfect transmitter CSI exploiting the duality result of [7]. In particular, we highlight the possibility to use a standard interference function based on the MMSE resulting from applying scalar equalizers in the vector MAC leading to a low complexity of the fixed-point iteration to compute the power allocation.

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2. SYSTEM MODEL

The upper subfigure of Fig. 1 depicts the BC model. The zero-mean data signal $s_k \in \mathbb{C}$ for user k, with $1 \leq k \leq K$ and $\mathrm{E}[|s_k|^2] = 1$, is precoded by $\boldsymbol{p}_k \in \mathbb{C}^N$, where K and N are the number of users and transmit antennas, respectively. The transmit signal propagates over the vector channel $\boldsymbol{h}_k \in \mathbb{C}^N$ and the additive Gaussian noise is $\eta_k \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_k^2)$. The estimate at the output of the scalar receiver $f_k \in \mathbb{C}$ reads as

$$\hat{s}_k = f_k \boldsymbol{h}_k^{\mathrm{H}} \textstyle\sum_{i=1}^{K} \boldsymbol{p}_i s_i + f_k \eta_k. \tag{1}$$

The data signals are mutually independent and also independent of the noise signals.

We assume that the transmitter does not perfectly know the CSI but has some partial CSI v and the parameters of the PDFs $\mathrm{f}_{\boldsymbol{h}_k|v}(\boldsymbol{h}_k|v)$ for all k are available. Contrarily, the receivers can employ the known full CSI. Thus, any meaningful equalizers are functions of the channel state (see [7]), e.g.,

$$f_{k,\mathrm{MMSE}} = \mathrm{argmin}_{f_k} \, \mathrm{E}\left[|s_k - \hat{s}_k|^2 \middle| \boldsymbol{h}_k \right]. \tag{2}$$

To highlight the dependence of the equalizers on the channel state, we use the notation $f_k(\boldsymbol{h}_k)$ in the following.

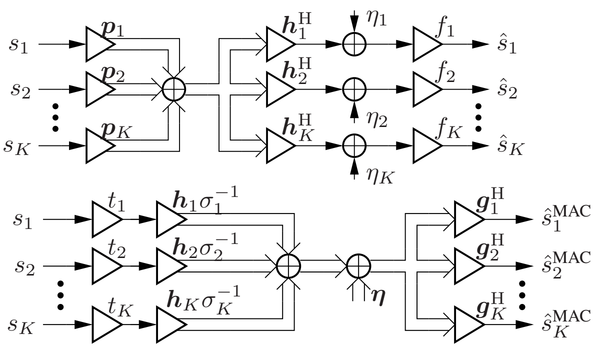

Fig. 1. Downlink and dual uplink

The transmitter, however, only has the partial CSI v. Therefore, the precoder design is based on the average MSE

$$\overline{\mathrm{MSE}}_k^{\mathrm{BC}} = \mathrm{E}\left[|s_k - \hat{s}_k|^2 \middle| v \right] = \mathrm{E}\left[1 - 2\Re\left\{ f_k(\boldsymbol{h}_k)\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{p}_k \right\} + \sum_{i=1}^K \left| f_k(\boldsymbol{h}_k)\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{p}_i \right|^2 + \sigma_k^2 \left| f_k(\boldsymbol{h}_k) \right|^2 \middle| v \right]. \quad (3)$$

The lower subfigure of Fig. 1 shows the MAC model. The kth precoder is $t_k(\boldsymbol{h}_k) \in \mathbb{C}$. The transmit signal propagates over the channel $\sigma_k^{-1}\boldsymbol{h}_k \in \mathbb{C}^N$. The received signal is perturbed by $\boldsymbol{\eta} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I})$ and filtered with the equalizer $\boldsymbol{g}_k \in \mathbb{C}^N$ to get the estimated symbol of user k, i.e., $\hat{s}_k^{\mathrm{MAC}} = \boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{x}$ with $\boldsymbol{x} = \sum_{i=1}^K \sigma_i^{-1}\boldsymbol{h}_i t_i(\boldsymbol{h}_i) s_i + \boldsymbol{\eta}$. Note that the MAC equalizers $\boldsymbol{g}_k$ depend on the partial CSI v whereas the MAC precoders $t_k(\boldsymbol{h}_k)$ are functions of the current channel state. Accordingly,

$$\overline{\mathrm{MSE}}_k^{\mathrm{MAC}} = \mathrm{E}\left[1 - 2\Re\left\{ \sigma_k^{-1}\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{h}_k t_k(\boldsymbol{h}_k) \right\} + \sum_{i=1}^K \sigma_i^{-2} \left| \boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{h}_i t_i(\boldsymbol{h}_i) \right|^2 + \|\boldsymbol{g}_k\|_2^2 \middle| v \right] \quad (4)$$

is the average MSE $\mathrm{E}[|s_k - \hat{s}_k^{\mathrm{MAC}}|^2 \,|\, v]$ in the MAC channel.

2.1. BC/MAC MSE Duality

We define the relationship between BC and MAC filters as [7]

$$\boldsymbol{p}_k = \alpha_k \boldsymbol{g}_k \quad \text{and} \quad f_k(\boldsymbol{h}_k) = \sigma_k^{-1}\alpha_k^{-1}t_k^*(\boldsymbol{h}_k) \quad (5)$$

with $\alpha_k \in \mathbb{R}^+$ and rewrite $\overline{\mathrm{MSE}}_k^{\mathrm{BC}}$ accordingly [cf. (3)], i.e.,

$$\overline{\mathrm{MSE}}_k^{\mathrm{BC}} = \mathrm{E}\left[1 - 2\Re\left\{ \sigma_k^{-1}t_k^*(\boldsymbol{h}_k)\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{g}_k \right\} + \alpha_k^{-2}|t_k(\boldsymbol{h}_k)|^2 + \sum_{i=1}^K \frac{\alpha_i^2}{\alpha_k^2}\sigma_k^{-2} \left| \boldsymbol{g}_i^{\mathrm{H}}\boldsymbol{h}_k t_k(\boldsymbol{h}_k) \right|^2 \middle| v \right].$$

By equating the last expression to (4), we get $\boldsymbol{\Gamma}\boldsymbol{a} = \boldsymbol{\varsigma}$, where $\boldsymbol{a} = [\alpha_1^2, \ldots, \alpha_K^2]^{\mathrm{T}}$ and with $\varsigma_i = \mathrm{E}[|t_i(\boldsymbol{h}_i)|^2 \,|\, v] \in \mathbb{R}_0^+$, we have $\boldsymbol{\varsigma} = [\varsigma_1, \ldots, \varsigma_K]^{\mathrm{T}}$. The entries of $\boldsymbol{\Gamma} \in \mathbb{R}^{K\times K}$ are

$$\gamma_{k,j} = \begin{cases} \sum_{i\neq k}\sigma_i^{-2}\,\mathrm{E}[|\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{h}_i t_i(\boldsymbol{h}_i)|^2 \,|\, v] + \|\boldsymbol{g}_k\|_2^2 & j = k \\ -\sigma_k^{-2}\,\mathrm{E}[|\boldsymbol{g}_j^{\mathrm{H}}\boldsymbol{h}_k t_k(\boldsymbol{h}_k)|^2 \,|\, v] & j \neq k. \end{cases}$$

Since $\boldsymbol{\Gamma}$ is diagonally dominant, $\boldsymbol{\Gamma}^{-1}$ exists. As $\boldsymbol{\Gamma}$ has positive diagonal and non-positive off-diagonal entries, $\boldsymbol{\Gamma}^{-1}$ has non-negative entries [5, 20] and the resulting α_k^2 are non-negative. Thus, $\alpha_k \in \mathbb{R}^+$ can always be found such that $\overline{\mathrm{MSE}}_k^{\mathrm{BC}} = \overline{\mathrm{MSE}}_k^{\mathrm{MAC}}, \forall k$. Left multiplying $\boldsymbol{\Gamma}\boldsymbol{a} = \boldsymbol{\varsigma}$ by the all-ones vector $\mathbf{1}^{\mathrm{T}}$ yields $\sum_{i=1}^K \|\boldsymbol{g}_i\|_2^2\alpha_i^2 = \sum_{i=1}^K \mathrm{E}[|t_i(\boldsymbol{h}_i)|^2 \,|\, v]$. Due to (5), we can infer that the same average transmit power is used in the BC as in the dual MAC.

The proof for the converse transform is analogous. For given BC filters, MAC filters achieving the same average MSEs with the same transmit power can be found [7].

3. PROBLEM FORMULATION

Due to Jensen's inequality and the concavity of $\log_2(\bullet)$, we have $\log_2(\mathrm{E}[x]) \geq \mathrm{E}[\log_2(x)]$. Since the instantaneous data rate can be expressed as $R = -\log_2(\mathrm{MMSE})$, we have that $\mathrm{E}[R] = \mathrm{E}[-\log_2(\mathrm{MMSE})] \geq -\log_2(\mathrm{E}[\mathrm{MMSE}])$. In other words, when ensuring an average MMSE, a minimum average rate is guaranteed, i.e., $\mathrm{E}[R_k \,|\, v] \geq -\log_2(\varepsilon_k)$ follows from $\overline{\mathrm{MMSE}}_k^{\mathrm{BC}} \leq \varepsilon_k$. To illustrate the quality of the lower bound, let the MMSE be beta distributed, i.e., MMSE $\sim \beta(a, b)$. Then, $-\log_2(\mathrm{E}[\mathrm{MMSE}]) = \log_2(1 + \frac{b}{a})$ and for positive integer a, b, it can be shown that $\mathrm{E}[R] \approx \log_2(1 + \frac{b}{a-1})$.

Our goal is to ensure minimum average rates. Based on above discussion, we circumvent the difficult optimization of the average rates and concentrate on the average MSE instead. We minimize the total transmit power under *Quality of Service* (QoS) constraints expressed as maximum MSEs ε_k, i.e.,

$$\min_{\{f_k(\boldsymbol{h}_k), \boldsymbol{p}_k\}_{k=1}^K} \sum_{i=1}^K \|\boldsymbol{p}_i\|^2 \quad \text{s.t.: } \forall k\text{: } \overline{\mathrm{MSE}}_k^{\mathrm{BC}} \leq \varepsilon_k \quad (6)$$

where the precoders $\boldsymbol{p}_k$ only depend on the partial CSI v. Note that this formulation ensures $\mathrm{E}[R_k \,|\, v] \geq -\log_2(\varepsilon_k), \forall k$. Moreover, the BC optimization (6) has the advantage that the computation of the equalizers is simple. From (2), we find

$$f_{k,\mathrm{MMSE}}(\boldsymbol{h}_k) = \left(\sigma_k^2 + \sum_{i=1}^K \left| \boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{p}_i \right|^2 \right)^{-1} \boldsymbol{p}_k^{\mathrm{H}}\boldsymbol{h}_k. \quad (7)$$

For the computation of the optimal precoders, however, a reformulation in the dual MAC is necessary, that is,

$$\min_{\{t_k(\boldsymbol{h}_k), \boldsymbol{g}_k\}_{k=1}^K} P_{\text{tx,MAC}} \quad \text{s.t.: } \forall k\text{: } \overline{\mathrm{MSE}}_k^{\mathrm{MAC}} \leq \varepsilon_k \quad (8)$$

with $P_{\text{tx,MAC}} = \sum_{i=1}^K \mathrm{E}[|t_i(\boldsymbol{h}_i)|^2 \,|\, v]$. The reformulation (8) leads to following optimal MAC equalizers (BC precoders)

$$\boldsymbol{g}_{k,\mathrm{MMSE}} = (\boldsymbol{R} + \mathbf{I})^{-1}\boldsymbol{\mu}_k \quad (9)$$

where we introduced $\boldsymbol{R} = \sum_{i=1}^K \sigma_i^{-2}\,\mathrm{E}[\boldsymbol{h}_i|t_i(\boldsymbol{h}_i)|^2\boldsymbol{h}_i^{\mathrm{H}} \,|\, v]$ and $\boldsymbol{\mu}_k = \sigma_k^{-1}\,\mathrm{E}[\boldsymbol{h}_k t_k(\boldsymbol{h}_k) \,|\, v]$. The two formulations (6) and (8) allow for a simple computation of the optimal equalizers but finding the precoders fulfilling the QoS constraints is difficult. Therefore, we propose to employ an *Alternating Optimization* (AO). The BC equalizers are found via (7) for given precoders $\boldsymbol{p}_k$ but the BC precoders including the power allocation are computed in the dual MAC for given $f_k(\boldsymbol{h}_k)$.

4. MAC SOLUTION FOR GIVEN BC EQUALIZERS

As can be seen in (9), it is necessary to compute the expectations $\boldsymbol{R}$ and $\boldsymbol{\mu}_i$ for $i = 1, \ldots, K$. We propose to perform the numerical integration by the Monte Carlo method. The M realizations resulting from the PDF $\mathrm{f}_{\boldsymbol{h}_k|v}(\boldsymbol{h}_k|v)$ are collected in $\boldsymbol{H}_k = \sigma_k^{-1}[\boldsymbol{h}_k^{(1)}, \ldots, \boldsymbol{h}_k^{(M)}]$. Likewise, $\boldsymbol{t}_k = [t_k(\boldsymbol{h}_k^{(1)}), \ldots, t_k(\boldsymbol{h}_k^{(M)})]^{\mathrm{T}}$ comprises the corresponding MAC precoders. In the AO procedure, the direction of $\boldsymbol{t}_k$, i.e., the dependence of the BC equalizers on the channel state, is left unchanged in the MAC step. However, the power allocation is updated in the MAC to fulfill the QoS constraints. To this end, we split off the power allocation $\xi_k = \|\boldsymbol{t}_k\|_2^2/M$, i.e., $\boldsymbol{t}_k = \sqrt{\xi_k}\boldsymbol{\tau}_k$ with $\|\boldsymbol{\tau}_k\|_2^2 = M$. For notational brevity, we use $\boldsymbol{T}_k = \mathrm{diag}(\tau_1, \ldots, \tau_M)$ such that $\boldsymbol{\tau}_k = \boldsymbol{T}_k\mathbf{1}$ with the all-ones vector $\mathbf{1}$. Accordingly, the MAC MSE reads as [cf. (4)]

$$\overline{\mathrm{MSE}_k^{\mathrm{MAC}}} = 1 - 2M^{-1}\sqrt{\xi_k}\Re\{\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_k\boldsymbol{T}_k\mathbf{1}\} + \tfrac{1}{M}\textstyle\sum_{i=1}^{K}\xi_i\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_i\boldsymbol{T}_i\boldsymbol{T}_i^{\mathrm{H}}\boldsymbol{H}_i^{\mathrm{H}}\boldsymbol{g}_k + \|\boldsymbol{g}_k\|_2^2. \tag{10}$$

The optimal equalizers $\boldsymbol{g}_{k,\mathrm{MMSE}}$ still have the form of (9) but $\boldsymbol{R} = \frac{1}{M}\sum_{i=1}^{K}\xi_i\boldsymbol{H}_i\boldsymbol{T}_i\boldsymbol{T}_i^{\mathrm{H}}\boldsymbol{H}_i^{\mathrm{H}}$ and $\boldsymbol{\mu}_k = \frac{1}{M}\sqrt{\xi_k}\boldsymbol{H}_k\boldsymbol{T}_k\mathbf{1}$. Next, the MAC power allocation $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_K]^{\mathrm{T}}$ is found.

4.1. Power Allocation via Interference Function

We discuss two interference functions. For the first and obvious one, $\boldsymbol{g}_{k,\mathrm{MMSE}}$ is implicitly applied. The computationally advantageous second one keeps the direction of $\boldsymbol{g}_k$ constant.

A) Matrix-Inversion Interference Function: Suppose that the optimal equalizers $\boldsymbol{g}_{k,\mathrm{MMSE}}$ [see (9)] are used. After applying the matrix inversion lemma, the resulting minimum MSE can be written as [cf. (10)]

$$\overline{\mathrm{MMSE}_k^{\mathrm{MAC}}} = \frac{1}{\xi_k}\mathbf{1}^{\mathrm{T}}\left(\frac{M}{\xi_k}\mathbf{I} + \boldsymbol{T}_k^{\mathrm{H}}\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{X}_k^{-1}\boldsymbol{H}_k\boldsymbol{T}_k\right)^{-1}\mathbf{1}$$

with $\boldsymbol{X}_k = \frac{1}{M}\sum_{i\neq k}\xi_i\boldsymbol{H}_i\boldsymbol{T}_i\boldsymbol{T}_i^{\mathrm{H}}\boldsymbol{H}_i^{\mathrm{H}} + \mathbf{I}$. Interpreting the term $I_k(\boldsymbol{\xi}) = \mathbf{1}^{\mathrm{T}}(\frac{M}{\xi_k}\mathbf{I} + \boldsymbol{T}_k^{\mathrm{H}}\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{X}_k^{-1}\boldsymbol{H}_k\boldsymbol{T}_k)^{-1}\mathbf{1}$ as interference, we have that $\overline{\mathrm{MMSE}_k^{\mathrm{MAC}}} = I_k(\boldsymbol{\xi})/\xi_k$. The MAC QoS problem reduces to [cf. (8)]

$$\min_{\boldsymbol{\xi}\geq\mathbf{0}}\mathbf{1}^{\mathrm{T}}\boldsymbol{\xi} \qquad \text{s.t.:} \quad \forall k\colon \varepsilon_k^{-1}I_k(\boldsymbol{\xi}) \leq \xi_k. \tag{11}$$

It can easily be shown that $\boldsymbol{I}(\boldsymbol{\xi}) = [I_1(\boldsymbol{\xi}), \ldots, I_K(\boldsymbol{\xi})]^{\mathrm{T}}$ is a standard interference function [18], i.e., we have positivity ($\boldsymbol{I}(\boldsymbol{\xi}) > \mathbf{0}$), monotonicity ($\boldsymbol{I}(\boldsymbol{\xi}) \geq \boldsymbol{I}(\boldsymbol{\xi}')$ for $\boldsymbol{\xi} \geq \boldsymbol{\xi}'$), and scalability ($z\boldsymbol{I}(\boldsymbol{\xi}) > \boldsymbol{I}(z\boldsymbol{\xi})$ for all $z > 1$). The inherent optimization w.r.t. the MAC equalizers $\boldsymbol{g}_k$ when employing $\boldsymbol{I}(\boldsymbol{\xi})$ is possible due to [18, Theorem 5] (see also [21]). Since $\boldsymbol{I}(\boldsymbol{\xi})$ is standard, the fixed point iteration $\boldsymbol{\xi}^{(\ell)} = \boldsymbol{E}^{-1}\boldsymbol{I}(\boldsymbol{\xi}^{(\ell-1)})$ with $\boldsymbol{E} = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_K)$ converges to the global optimum of the power minimization (11) and delivers the optimum power allocation $\boldsymbol{\xi}_{\mathrm{opt}}$ and MAC equalizers $\boldsymbol{g}_{\mathrm{opt},k}$ for given MAC beamformers $\boldsymbol{T}_k, k \in \{1, \ldots, K\}$ (see [21]).

B) Scalar-Inversion Interference Function: To save computational complexity by avoiding the $2K$ inversions in the definition of $\boldsymbol{I}(\boldsymbol{\xi})$, the MAC equalizers $\boldsymbol{g}_k$ resulting from the BC-to-MAC transform are kept fixed. To allow for an adaptation of the equalizers, additional scalar equalizers r_k are introduced. Replacing $\boldsymbol{g}_k$ by $r_k\boldsymbol{g}_k$ in (10) leads to

$$\overline{\mathrm{MSE}_k^{\mathrm{MAC}}} = 1 - 2M^{-1}\Re\left\{r_k^*\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_k\boldsymbol{T}_k\mathbf{1}\sqrt{\xi_k}\right\} + \frac{1}{M}|r_k|^2\textstyle\sum_{i=1}^{K}\xi_i\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_i\boldsymbol{T}_i\boldsymbol{T}_i^{\mathrm{H}}\boldsymbol{H}_i^{\mathrm{H}}\boldsymbol{g}_k + |r_k|^2\|\boldsymbol{g}_k\|_2^2. \tag{12}$$

The k-th MMSE optimal scalar receiver is given by

$$r_{k,\mathrm{MMSE}} = \frac{\frac{1}{M}\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_k\boldsymbol{T}_k\mathbf{1}\sqrt{\xi_k}}{\frac{1}{M}\sum_{i=1}^{K}\xi_i\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_i\boldsymbol{T}_i\boldsymbol{T}_i^{\mathrm{H}}\boldsymbol{H}_i^{\mathrm{H}}\boldsymbol{g}_k + \|\boldsymbol{g}_k\|_2^2}. \tag{13}$$

Substituting $r_{k,\mathrm{MMSE}}$ in (12) gives $\overline{\mathrm{MMSE}_{k,\mathrm{scalar}}^{\mathrm{MAC}}}$. With

$$y_k(\boldsymbol{\xi}) = \frac{1}{M}\sum_{i=1}^{K}\xi_i\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_i\boldsymbol{T}_i\boldsymbol{T}_i^{\mathrm{H}}\boldsymbol{H}_i^{\mathrm{H}}\boldsymbol{g}_k - \frac{\xi_k}{M^2}\left|\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_k\boldsymbol{T}_k\mathbf{1}\right|^2$$

and $x_k(\boldsymbol{\xi}) = \|\boldsymbol{g}_k\|_2^2 + y_k(\boldsymbol{\xi})$, the minimum MSE reads as

$$\overline{\mathrm{MMSE}_{k,\mathrm{scalar}}^{\mathrm{MAC}}} = \frac{1}{\xi_k}\left(\frac{1}{\xi_k} + \frac{\left|\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_k\boldsymbol{T}_k\mathbf{1}\right|^2}{M^2x_k(\boldsymbol{\xi})}\right)^{-1}. \tag{14}$$

For diagonal $\boldsymbol{D}$, $\boldsymbol{a}^{\mathrm{H}}\boldsymbol{D}^2\boldsymbol{a} - \frac{1}{M}|\boldsymbol{a}^{\mathrm{H}}\boldsymbol{D}\mathbf{1}|^2 = \boldsymbol{a}^{\mathrm{H}}\boldsymbol{D}\boldsymbol{\Pi}\boldsymbol{D}\boldsymbol{a} > 0$ with the projector $\boldsymbol{\Pi} = \mathbf{I} - \frac{1}{M}\mathbf{1}\mathbf{1}^{\mathrm{T}}$. Thus, $x_k(\boldsymbol{\xi}) > 0$. The QoS power allocation problem can be written as [cf. (8)]

$$\min_{\boldsymbol{\xi}\geq\mathbf{0}}\mathbf{1}^{\mathrm{T}}\boldsymbol{\xi} \qquad \text{s.t.:} \quad \forall k\colon \varepsilon_k^{-1}J_k(\boldsymbol{\xi}) \leq \xi_k. \tag{15}$$

For $\overline{\mathrm{MMSE}_{k,\mathrm{scalar}}^{\mathrm{MAC}}} = \frac{J_k(\boldsymbol{\xi})}{\xi_k}$, the interference of user k is set to

$$J_k(\boldsymbol{\xi}) = \left(\frac{1}{\xi_k} + \frac{1}{M^2}\frac{1}{x_k(\boldsymbol{\xi})}\left|\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_k\boldsymbol{T}_k\mathbf{1}\right|^2\right)^{-1}. \tag{16}$$

Collecting the interferences in $\boldsymbol{J}(\boldsymbol{\xi}) = [J_1(\boldsymbol{\xi}), \ldots, J_K(\boldsymbol{\xi})]^{\mathrm{T}}$ gives a standard interference function. Positivity of $\boldsymbol{J}(\boldsymbol{\xi})$ follows from $\boldsymbol{\xi} \geq \mathbf{0}$ and $x_k > 0$. Monotonicity can be seen from the property of x_k to be monotonically increasing in $\boldsymbol{\xi}$. Finally, we have $zx_k(\boldsymbol{\xi}) > x_k(z\boldsymbol{\xi})$ for $z > 1$ and thus, $zJ_k(\boldsymbol{\xi}) > J_k(z\boldsymbol{\xi})$ manifesting scalability. As $\boldsymbol{J}(\boldsymbol{\xi})$ is a standard interference function, the iteration $\boldsymbol{\xi}^{(\ell)} = \boldsymbol{E}^{-1}\boldsymbol{J}(\boldsymbol{\xi}^{(\ell-1)})$ with $\boldsymbol{E} = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_K)$ converges to the global optimum of (15), i.e., $\boldsymbol{\xi}_{\mathrm{opt}}$ and $r_{\mathrm{opt},k}$ for given $\boldsymbol{g}_k$ and $\boldsymbol{T}_k$ with $k \in \{1, \ldots, K\}$. Comparing the expression (16) for $J_k(\boldsymbol{\xi})$ to that of $I_k(\boldsymbol{\xi})$ illustrates the simplicity of $J_k(\boldsymbol{\xi})$.

Algorithm 1 Power Minimization

1: $l \leftarrow 0$, random init.: $\boldsymbol{p}_k^{(0)}$ and $\boldsymbol{h}_k^{(m)} \sim \mathrm{f}_{\boldsymbol{h}_k|v}(\boldsymbol{h}_k|v), \forall k, m$
2: **repeat**
3: $\quad l \leftarrow l+1$, execute commands for all $k \in \{1,\ldots,K\}$
4: $\quad$ **for** $m = 1$ to M **do**
5: $\quad\quad f_k^{(l,m)} \leftarrow (\sum_{i=1}^{K} |\boldsymbol{h}_k^{(m),\mathrm{H}} \boldsymbol{p}_i^{(l-1)}|^2 + \sigma_k^2)^{-1} \boldsymbol{p}_k^{(l-1),\mathrm{H}} \boldsymbol{h}_k^{(m)}$
6: $\quad$ **end for**
7: $\quad \boldsymbol{t}_k^{(l)} \leftarrow$ BC-to-MAC conversion (see Section 2.1)
8: $\quad \xi_k^{(l+1)} \leftarrow \frac{1}{\varepsilon_k} J_k(\boldsymbol{\xi}^{(l)})$
9: $\quad \boldsymbol{t}_k^{(l+1)} \leftarrow \boldsymbol{\tau}_k^{(l)} \sqrt{\xi_k^{(l+1)}}$
10: $\quad \boldsymbol{g}_k^{(l+1)} \leftarrow$ update MAC receiver using (9)
11: $\quad \boldsymbol{p}_k^{(l+1)} \leftarrow$ MAC to BC conversion (see Section 2.1)
12: **until** $\left|\boldsymbol{\xi}^{(l+1)} - \boldsymbol{\xi}^{(l)}\right| \leq \delta$

Note that the matrix vector products necessary in (16) have already been computed during the BC-to-MAC transform. Hence, only simple scalar operations have to be performed. For $I_k(\boldsymbol{\xi})$, however, two matrix inversions per user have to be computed per step of the fixed point iteration.

4.2. Equivalence of Interference Functions

From (9), we have $\boldsymbol{g}_{k,\mathrm{MMSE}} = (\boldsymbol{R} + \mathbf{I})^{-1} \boldsymbol{\mu}_k$ for the optimal MAC equalizers with $\boldsymbol{R} = \frac{1}{M} \sum_{i=1}^{K} \xi_i \boldsymbol{H}_i \boldsymbol{T}_i \boldsymbol{T}_i^{\mathrm{H}} \boldsymbol{H}_i^{\mathrm{H}}$ and $\boldsymbol{\mu}_k = \frac{1}{M} \sqrt{\xi_k} \boldsymbol{H}_k \boldsymbol{T}_k \mathbf{1}$. Substituting $\boldsymbol{g}_{k,\mathrm{MMSE}}$ in $x_k(\boldsymbol{\xi})$ gives

$$x_k(\boldsymbol{\xi}) = \boldsymbol{\mu}_k^{\mathrm{H}} (\boldsymbol{R} + \mathbf{I})^{-1} \boldsymbol{\mu}_k - \left(\boldsymbol{\mu}_k^{\mathrm{H}} (\boldsymbol{R} + \mathbf{I})^{-1} \boldsymbol{\mu}_k\right)^2 .$$

The MMSE with scalar equalizer $r_{k,\mathrm{MMSE}}$ can therefore be rewritten as [cf. (14)]

$$\overline{\mathrm{MMSE}_{k,\mathrm{scalar}}^{\mathrm{MAC}}} = 1 - \frac{1}{M^2} \xi_k \boldsymbol{\mu}_k^{\mathrm{H}} (\boldsymbol{R} + \mathbf{I})^{-1} \boldsymbol{\mu}_k .$$

Applying the matrix inversion lemma leads to the conclusion that $\overline{\mathrm{MMSE}_{k,\mathrm{scalar}}^{\mathrm{MAC}}} = \overline{\mathrm{MMSE}_k^{\mathrm{MAC}}}$ if $\boldsymbol{g}_k = \boldsymbol{g}_{k,\mathrm{MMSE}}$. Thus, the two interference functions lead to the same power allocation in each step if the equalizers are updated in every step of the fixed point iteration with the scalar interference function.

5. ALGORITHMIC SOLUTION

The pseudocode in Algorithm 1 solves the power minimization problem (6). In every loop, the BC equalizers are updated in line 5. After the BC-to-MAC transform, the MAC power allocation is recomputed based on the interference function $\boldsymbol{J}(\boldsymbol{\xi})$ [see (16)] in line 8. The MAC equalizers are updated in line 10. Due to the MAC-to-BC transform in line 11, this corresponds to an update of the BC precoders. Note that Algorithm 1 is performed at the BC transmitter based on the partial CSI. No computations are necessary at the receivers.

Every step of Algorithm 1 either reduces the transmit power or the average MSEs (without changing the transmit power). Due to the existence of a unique minimum of (6), this property implies that the power converges. Note that the precoders and equalizers are not unique, e.g., weighting $\boldsymbol{p}_k$ with $\exp(\mathrm{j}\,\varphi)$ and $t_k(\boldsymbol{h}_k)$ with $\exp(-\mathrm{j}\,\varphi)$ does neither influence the power nor the average MSEs. Nevertheless, we observed that also the filters always converge.

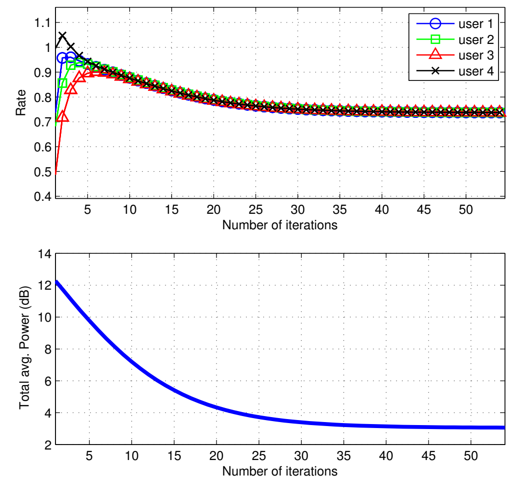

Fig. 2. Example of algorithm execution.

6. SIMULATIONS

We present the results of a simulation for $N = 4$ transmit antennas and $K = 4$ users, considering $\sigma_k^2 = 1, \forall k$. The upper subfigure of Fig. 2 shows the average rates vs. the number of iterations, while the lower subfigure shows the total transmit power vs. the number of iterations. The threshold δ is set to 10^{-4} and the result is the mean of 4000 channel realizations. The partial CSI v is translated into the channel first and second order moments, i.e., $\forall k\colon\ \mathrm{E}[\boldsymbol{h}_k|v] = \boldsymbol{u}_k$ with $u_{k,l} = \mathrm{e}^{\mathrm{j}(l-1)\varphi_k}$ and $\varphi_k \sim \mathcal{U}(0, 2\pi)$, and $\forall k\colon \boldsymbol{C}_{\boldsymbol{h}_k|v} = \mathbf{I}_N$, respectively. The average MMSE targets are $\varepsilon_k = 0.6, \forall k$. That is, $\mathrm{E}[R_k] \geq -\log_2(0.6) = 0.7370, \forall k$. As we can see in Fig. 2, these rate targets are met in a very good approximation after convergence although the algorithm is based on a lower bound of the average rates.

7. CONCLUSIONS

We proposed an algorithm for the power minimization in the vector BC under minimum ergodic rate constraints via imposing conservative average MSE constraints. Using the average MSE BC/MAC duality, the equalizer filters are updated in each iteration and the transmit power is minimized by means of standard interference functions. As the problem formulation is meaningless for infeasible targets, the characterization of the feasible region is a possible future work.

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