Power Minimization and QoS Feasibility Region in the Multiuser MIMO Broadcast Channel with Imperfect CSI

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***Abstract*—The aim of this work is to jointly achieve individual rate requirements and minimum total transmit power in a *Multi User Multiple Input Multiple Output* (MU-MIMO) *Broadcast Channel* (BC). Data streams are transmitted from a multi-antenna base station to several independent and non-cooperative multi-antenna users. Perfect *Channel-State-Information* (CSI) is known at the receivers and it is fed back to the transmitter, where partial CSI is used for the design of linear transmit filters. Employing a duality between *Multiple Access Channel* (MAC) and BC w.r.t. the average *Mean Square Error* (MSE) and identifying *standard* interference functions, we propose an algorithmic joint solution for the transmit filter design and the power allocation. Additionally, we describe the feasibility region in the average MMSE domain. This result allows for checking the convergence of the algorithm for given *Quality of Service* (QoS) constraints.**

***Keywords*—MIMO, Multiuser, QoS, Feasibility, Imperfect CSI**

I. Introduction

Dualities between the MAC and the BC have been studied in several previous works, e.g., dualities w.r.t. *Signal-to-Interference-and-Noise Ratio* (SINR) [1], [2], [5], rate [3], [4], and MSE [6]–[8]. Using several types of dualities, regions of the MAC and the BC were confirmed to be identical under sum power constraints. For example, the SINR duality of the vector BC and the vector MAC [1] is used in SINR balancing algorithms (e.g. [2]). The coincidence of the MSE regions of the MIMO MAC and the MIMO BC has also been proven in [6], [7]. Finally, the rate regions of the MAC and the BC have been observed to be the same with and without non-linear interference cancellation under Gaussian signaling for both single antenna and multiple antenna scenarios [3]–[5].

The mentioned dualities provide conversion formulas to switch from one domain to the other. These dualities (except the result in [8]) and the MMSE balancing solution via interference functions of [9] are based not only on the assumption of perfect CSI at the receivers but also at the transmitter in the BC. In a practical scenario, however, the transmit CSI is imperfect, e.g., when the transmitter obtains the CSI via a limited rate feedback channel.

Note that the optimization of the linear precoding operation based on the data rates is difficult for imperfect CSI. Therefore, we rely on the average MSE to end up with the optimization of the average rates lower bounds (see Section III). The proof of the duality between the BC and the MAC w.r.t. the average MSE conserving the average total transmit power was shown in [8]. Such a duality allows us to build a *standard* interference function [10] to find the optimal transceivers and power allocation by a fixed-point iteration. Thus, we proposed an algorithmic solution for the *Multi User Multiple Input Single Output* (MU-MISO) in [18] which is now extended to the MIMO case. Additionally, we describe a method to guarantee the convergence of the algorithm since it depends on whether the QoS constraints are feasible or not. There are several works concerning feasibility (e.g. [2], [16], [17]) but, again, they are based on the perfect CSI assumption.

We show that the QoS feasibility region in the average MMSE domain is a polytope. Any point inside the polytope is achievable and the boundaries describe the separation between feasibility and infeasibility. As the average MMSE can be translated to a lower bound to the rate, we end up with a sufficient feasibility condition for the rates under imperfect transmit CSI.

II. System model

The upper subfigure of Fig. 1 depicts the BC model considered in this work. The zero-mean data signal $s_k \in \mathbb{C}$ for user k, with $1 \leq k \leq K$ and $\mathrm{E}[|s_k|^2] = 1$, is precoded by $\boldsymbol{p}_k \in \mathbb{C}^N$, where K and N are the number of users and transmit antennas, respectively. The transmit signal propagates over a MIMO channel $\boldsymbol{H}_k \in \mathbb{C}^{N\times R}$, with R being the number of receiver antennas. The additive Gaussian noise in the MIMO channel is $\boldsymbol{\eta}_k \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{C}_{\eta_k})$. The data signals are mutually independent and also independent of the noise. The linear equalizer $\boldsymbol{f}_k \in \mathbb{C}^R$ provides the estimates of the data symbols

$$\hat{s}_k = \boldsymbol{f}_k^{\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}} \sum_{i=1}^{K} \boldsymbol{p}_i s_i + \boldsymbol{f}_k^{\mathrm{H}} \boldsymbol{\eta_k}. \tag{1}$$

In this work, we consider that the transmitter does not have a perfect knowledge of the CSI but a partial one that is modeled through v. We assume the conditional PDFs $f_{\boldsymbol{H}_k|v}(\boldsymbol{H}_k|v)$ are available for all k. Contrarily, the receivers can employ the known full CSI. Hence, any meaningful equalizers are functions of the channel state (see [8]), e.g.,

$$\boldsymbol{f}_{k,\mathrm{MMSE}} = \mathrm{argmin}_{\boldsymbol{f}_k} \mathrm{E}\left[|s_k - \hat{s}_k|^2 \middle| \boldsymbol{H}_k \right]. \tag{2}$$

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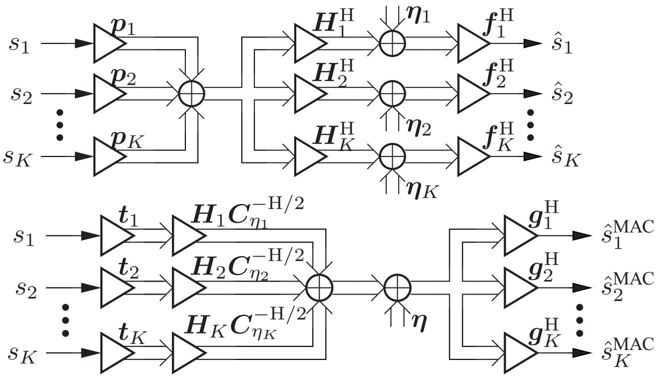

Fig. 1: Downlink and dual uplink

To highlight the dependence of the receivers on the channel state, we use the notation $\boldsymbol{f}_k^{\mathrm{H}}(\boldsymbol{H}_k)$ in the following.

Recall that the transmitter only has the partial CSI v. Hence, the precoder design is based on the average MSE

$$\overline{\mathrm{MSE}}_k^{\mathrm{BC}} = \mathrm{E}[|s_k - \hat{s}_k|^2 \,|\, v] = \mathrm{E}\left[1 - 2\Re\left\{\boldsymbol{f}_k^{\mathrm{H}}(\boldsymbol{H}_k)\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{p}_k\right\}\right.$$
$$\left. + \boldsymbol{f}_k^{\mathrm{H}}(\boldsymbol{H}_k)\boldsymbol{C}_{\eta_k}\boldsymbol{f}_k(\boldsymbol{H}_k) + \sum\nolimits_{i=1}^{K} \left|\boldsymbol{f}_k^{\mathrm{H}}(\boldsymbol{H}_k)\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{p}_i\right|^2 \right| v\Big]. \quad (3)$$

The lower subfigure of Fig. 1 shows the model of the dual MAC. The kth precoder is $\boldsymbol{t}_k(\boldsymbol{H}_k) \in \mathbb{C}^R$. The transmit signal propagates over the channel $\boldsymbol{H}_k\boldsymbol{C}_{\eta_k}^{-\mathrm{H}/2} \in \mathbb{C}^{N\times R}$. The received signal is perturbed by $\boldsymbol{\eta} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_N)$ and filtered with the receiver $\boldsymbol{g}_k \in \mathbb{C}^N$ to get the estimated symbol of user k, i.e., $\hat{s}_k^{\mathrm{MAC}} = \boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{x}$, with $\boldsymbol{x} = \sum_{i=1}^{K} \boldsymbol{H}_i\boldsymbol{C}_{\eta_i}^{-\mathrm{H}/2}\boldsymbol{t}_i(\boldsymbol{H}_i)s_i + \boldsymbol{\eta}$. Note that the MAC receivers $\boldsymbol{g}_k$ depend on the partial CSI v, whereas the MAC precoders $\boldsymbol{t}_k(\boldsymbol{H}_k)$ are functions of the current channel state. Accordingly,

$$\overline{\mathrm{MSE}}_k^{\mathrm{MAC}} = \mathrm{E}\left[1 - 2\Re\left\{\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_k\boldsymbol{C}_{\eta_k}^{-\mathrm{H}/2}\boldsymbol{t}_k(\boldsymbol{H}_k)\right\}\right. \quad (4)$$
$$\left. + \sum\nolimits_{i=1}^{K} \left|\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_i\boldsymbol{C}_{\eta_i}^{-\mathrm{H}/2}\boldsymbol{t}_i(\boldsymbol{H}_i)\right|^2 + \|\boldsymbol{g}_k\|_2^2 \right| v\Big]$$

is the average MSE, $\mathrm{E}[|s_k - \hat{s}_k^{\mathrm{MAC}}|^2 \,|\, v]$, in the MAC channel.

A. BC/MAC MSE Duality

We define the relationship between the BC and the MAC filters as [8]

$$\boldsymbol{p}_k = \alpha_k \boldsymbol{g}_k \quad \text{and} \quad \boldsymbol{f}_k(\boldsymbol{H}_k) = \alpha_k^{-1}\boldsymbol{C}_{\eta_k}^{-\mathrm{H}/2}\boldsymbol{t}_k(\boldsymbol{H}_k) \quad (5)$$

with $\alpha_k \in \mathbb{R}^+$ and rewrite $\overline{\mathrm{MSE}}_k^{\mathrm{BC}}$ accordingly [cf. (3)], i.e.,

$$\overline{\mathrm{MSE}}_k^{\mathrm{BC}} = \mathrm{E}\left[1 - 2\Re\left\{\boldsymbol{t}_k^{\mathrm{H}}(\boldsymbol{H}_k)\boldsymbol{C}_{\eta_k}^{-1/2}\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{g}_k\right\}\right.$$
$$\left. + \alpha_k^{-2}\|\boldsymbol{t}_k(\boldsymbol{H}_k)\|_2^2 + \sum\nolimits_{i=1}^{K} \frac{\alpha_i^2}{\alpha_k^2}\left|\boldsymbol{g}_i^{\mathrm{H}}\boldsymbol{H}_k\boldsymbol{C}_{\eta_k}^{-\mathrm{H}/2}\boldsymbol{t}_k(\boldsymbol{H}_k)\right|^2 \right| v\Big].$$

By equating the last expression to (4), we get $\boldsymbol{\Psi}\boldsymbol{a} = \boldsymbol{\varsigma}$, where $\boldsymbol{a} = [\alpha_1^2, \ldots, \alpha_K^2]^{\mathrm{T}}$ and with $\varsigma_i = \mathrm{E}[\|\boldsymbol{t}_i(\boldsymbol{H}_i)\|^2 \,|\, v] \in \mathbb{R}_0^+$, we have $\boldsymbol{\varsigma} = [\varsigma_1, \ldots, \varsigma_K]^{\mathrm{T}}$. The entries of $\boldsymbol{\Psi} \in \mathbb{R}^{K\times K}$ are

$$\psi_{k,j} = \begin{cases} \sum_{i\neq k} \mathrm{E}[|\boldsymbol{g}_k^{\mathrm{H}}\boldsymbol{H}_i\boldsymbol{C}_{\eta_i}^{-\mathrm{H}/2}\boldsymbol{t}_i(\boldsymbol{H}_i)|^2 \,|\, v] + \|\boldsymbol{g}_k\|_2^2 & j = k \\ -\mathrm{E}[|\boldsymbol{g}_j^{\mathrm{H}}\boldsymbol{H}_k\boldsymbol{C}_{\eta_k}^{-\mathrm{H}/2}\boldsymbol{t}_k(\boldsymbol{H}_k)|^2 \,|\, v] & j \neq k. \end{cases}$$

Note that $\boldsymbol{\Psi}$ is non-singular because it is diagonally dominant. Additionally, $\boldsymbol{\Psi}$ has positive diagonal and non-positive off-diagonal entries. Thus, $\boldsymbol{\Psi}^{-1}$ has non-negative entries [7], [13] and the resulting α_k^2 are non-negative. In other words, we can always find $\alpha_k \in \mathbb{R}^+$ such that $\overline{\mathrm{MSE}}_k^{\mathrm{BC}} = \overline{\mathrm{MSE}}_k^{\mathrm{MAC}}$, $\forall k$. Note that $\sum_{i=1}^{K} \|\boldsymbol{g}_i\|_2^2\alpha_i^2 = \sum_{i=1}^{K} \mathrm{E}[\|\boldsymbol{t}_i(\boldsymbol{H}_i)\|_2^2 \,|\, v]$, which results from left multiplying $\boldsymbol{\Psi}\boldsymbol{a} = \boldsymbol{\varsigma}$ by the all-ones vector $\mathbf{1}^{\mathrm{T}}$. Due to (5), we can infer that the same average transmit power is used in the BC as in the dual MAC.

The proof for the converse transform is analogous. For given BC filters, MAC filters achieving the same average MSEs with the same average transmit power can be found [8].

III. Problem formulation

Due to Jensen's inequality and the concavity of $\log_2(\bullet)$, we have $\log_2(\mathrm{E}[x]) \geq \mathrm{E}[\log_2(x)]$. Since the instantaneous data rate can be expressed as $R = -\log_2(\mathrm{MMSE})$, we have that $\mathrm{E}[R] = \mathrm{E}[-\log_2(\mathrm{MMSE})] \geq -\log_2(\mathrm{E}[\mathrm{MMSE}])$. In other words, when ensuring an average MMSE, a minimum average rate is guaranteed, i.e., $\mathrm{E}[R_k \,|\, v] \geq -\log_2(\varepsilon_k)$ follows from $\overline{\mathrm{MMSE}}_k^{\mathrm{BC}} \leq \varepsilon_k$.

Our goal is to ensure minimum average rates. Based on the above discussion, we circumvent the difficult optimization of the average rates and focus on the average MSE instead. We minimize the total transmit power under QoS constraints expressed as maximum MSEs, ε_k, i.e.,

$$\min_{\{\boldsymbol{f}_k(\boldsymbol{H}_k),\, \boldsymbol{p}_k\}_{k=1}^K} \sum_{i=1}^{K} \|\boldsymbol{p}_i\|^2 \qquad \text{s.t.:} \quad \forall k\colon \overline{\mathrm{MSE}}_k^{\mathrm{BC}} \leq \varepsilon_k \quad (6)$$

where the precoders $\boldsymbol{p}_k$ depend on the partial CSI v. This formulation is conservative as it ensures $\mathrm{E}[R_k \,|\, v] \geq -\log_2(\varepsilon_k)$ for all k. Note that the BC optimization (6) has the advantage that the computation of the optimal equalizers depending on the precoders is simple. From (2), we find

$$\boldsymbol{f}_{k,\mathrm{MMSE}}(\boldsymbol{H}_k) = \left(\boldsymbol{C}_{\eta_k} + \sum\nolimits_{i=1}^{K} \boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{p}_i\boldsymbol{p}_i^{\mathrm{H}}\boldsymbol{H}_k\right)^{-1} \boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{p}_k. \quad (7)$$

For the computation of the optimal precoders, however, a reformulation in the dual MAC is necessary, i.e.,

$$\min_{\{\boldsymbol{t}_k(\boldsymbol{H}_k),\, \boldsymbol{g}_k\}_{k=1}^K} P_{\mathrm{tx,MAC}} \qquad \text{s.t.:} \quad \forall k\colon \overline{\mathrm{MSE}}_k^{\mathrm{MAC}} \leq \varepsilon_k \quad (8)$$

with $P_{\mathrm{tx,MAC}} = \sum_{i=1}^{K} \mathrm{E}[||\boldsymbol{t}_i(\boldsymbol{H}_i)||_2^2 \,|\, v]$. The reformulation (8) leads to the following optimal MAC receivers, i.e., the BC precoders [see (5)]

$$\boldsymbol{g}_{k,\mathrm{MMSE}} = (\boldsymbol{R} + \mathbf{I}_N)^{-1}\boldsymbol{\mu}_k \quad (9)$$

where we introduced $\boldsymbol{\mu}_k = \mathrm{E}[\boldsymbol{H}_k\boldsymbol{C}_{\eta_k}^{-\mathrm{H}/2}\boldsymbol{t}_k(\boldsymbol{H}_k) \,|\, v]$ and $\boldsymbol{R} = \sum_{i=1}^{K} \mathrm{E}[\boldsymbol{H}_i\boldsymbol{C}_{\eta_i}^{-\mathrm{H}/2}\boldsymbol{t}_i(\boldsymbol{H}_i)\boldsymbol{t}_i^{\mathrm{H}}(\boldsymbol{H}_i)\boldsymbol{C}_{\eta_i}^{-1/2}\boldsymbol{H}_i^{\mathrm{H}} \,|\, v]$.

The two formulations (6) and (8) allow for a simple computation of the optimal receivers although the precoders fulfilling the QoS constraints are difficult to find. Therefore, we propose to employ an *Alternating Optimization* (AO) where BC receivers are found via (7) for given precoders $\boldsymbol{p}_k$ and the BC precoders (including the power allocation) are computed in the dual MAC for given $\boldsymbol{f}_k(\boldsymbol{H}_k)$ (see Section V).

IV. MAC Solution for Given BC Receivers

As can be seen from (9), it is necessary to compute the expectations $\boldsymbol{R}$ and $\boldsymbol{\mu}_i$ for $i = 1, \ldots, K$. We propose to perform the numerical integration by the Monte Carlo method. To this end, the M realizations $\boldsymbol{H}_k^{(1)}, \ldots, \boldsymbol{H}_k^{(M)}$ resulting from the PDF $f_{\boldsymbol{H}_k|v}(\boldsymbol{H}_k|v)$ are collected in $\boldsymbol{\Gamma}_k = [\boldsymbol{H}_k^{(1)}\boldsymbol{C}_{\eta_k}^{-\mathrm{H}/2}, \ldots, \boldsymbol{H}_k^{(M)}\boldsymbol{C}_{\eta_k}^{-\mathrm{H}/2}]$. In the AO procedure, the dependence of the BC receivers on the channel state is left unchanged in the MAC step. However, the power allocation is updated in the MAC to fulfill the QoS constraints. Therefore, we split off the average power allocation $\xi_k = \frac{1}{M}\sum_{i=1}^M \|\boldsymbol{t}_k^{(i)}\|_2^2$, i.e., $\boldsymbol{t}_k^{(i)} = \sqrt{\xi_k}\boldsymbol{\tau}_k^{(i)}$ with $\frac{1}{M}\sum_{i=1}^M \|\boldsymbol{\tau}_k^{(i)}\|_2^2 = 1, \forall i$. For notational brevity, we use $\boldsymbol{T}_k = \mathrm{blockdiag}(\boldsymbol{\tau}_k^{(1)}, \ldots, \boldsymbol{\tau}_k^{(M)})$ such that $\boldsymbol{T}_k\mathbf{1} = [\boldsymbol{\tau}_k^{(1),\mathrm{T}}, \ldots, \boldsymbol{\tau}_k^{(M),\mathrm{T}}]^\mathrm{T}$, with the all-ones vector $\mathbf{1}$. Accordingly, the MAC MSE reads as [cf. (4)]

$$\overline{\mathrm{MSE}}_k^{\mathrm{MAC}} = 1 - 2M^{-1}\sqrt{\xi_k}\Re\{\boldsymbol{g}_k^\mathrm{H}\boldsymbol{\Gamma}_k\boldsymbol{T}_k\mathbf{1}\} + \frac{1}{M}\sum_{i=1}^K \xi_i \boldsymbol{g}_k^\mathrm{H}\boldsymbol{\Gamma}_i\boldsymbol{T}_i\boldsymbol{T}_i^\mathrm{H}\boldsymbol{\Gamma}_i^\mathrm{H}\boldsymbol{g}_k + \|\boldsymbol{g}_k\|_2^2. \tag{10}$$

The optimal receivers $\boldsymbol{g}_{k,\mathrm{MMSE}}$ still have the form of (9) but $\boldsymbol{R} = \frac{1}{M}\sum_{i=1}^K \xi_i\boldsymbol{\Gamma}_i\boldsymbol{T}_i\boldsymbol{T}_i^\mathrm{H}\boldsymbol{\Gamma}_i^\mathrm{H}$ and $\boldsymbol{\mu}_k = \frac{1}{M}\sqrt{\xi_k}\boldsymbol{\Gamma}_k\boldsymbol{T}_k\mathbf{1}$. In the following, we show a strategy to find the MAC power allocation $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_K]^\mathrm{T} \geq \mathbf{0}$.

A. Power Allocation via Interference Functions

The MAC receivers $\boldsymbol{g}_k$ resulting from the BC-to-MAC transform are kept fixed. To allow for an adaptation of the equalizers during the power update, additional scalar receivers r_k are introduced. Replacing $\boldsymbol{g}_k$ by $r_k\boldsymbol{g}_k$ in (10) leads to

$$\overline{\mathrm{MSE}}_k^{\mathrm{MAC}} = 1 - 2M^{-1}\Re\left\{r_k^*\boldsymbol{g}_k^\mathrm{H}\boldsymbol{\Gamma}_k\boldsymbol{T}_k\mathbf{1}\sqrt{\xi_k}\right\} + \frac{1}{M}|r_k|^2\sum_{i=1}^K \xi_i\boldsymbol{g}_k^\mathrm{H}\boldsymbol{\Gamma}_i\boldsymbol{T}_i\boldsymbol{T}_i^\mathrm{H}\boldsymbol{\Gamma}_i^\mathrm{H}\boldsymbol{g}_k + |r_k|^2\|\boldsymbol{g}_k\|_2^2. \tag{11}$$

The MMSE optimal scalar receiver is given by

$$r_{k,\mathrm{MMSE}} = \frac{\frac{1}{M}\boldsymbol{g}_k^\mathrm{H}\boldsymbol{\Gamma}_k\boldsymbol{T}_k\mathbf{1}\sqrt{\xi_k}}{\frac{1}{M}\sum_{i=1}^K \xi_i\boldsymbol{g}_k^\mathrm{H}\boldsymbol{\Gamma}_i\boldsymbol{T}_i\boldsymbol{T}_i^\mathrm{H}\boldsymbol{\Gamma}_i^\mathrm{H}\boldsymbol{g}_k + \|\boldsymbol{g}_k\|_2^2}. \tag{12}$$

Substituting $r_{k,\mathrm{MMSE}}$ in (11) gives $\overline{\mathrm{MMSE}}_{k,\mathrm{scalar}}^{\mathrm{MAC}}$. With

$$y_k(\boldsymbol{\xi}) = \frac{1}{M}\sum_{i=1}^K \xi_i\boldsymbol{g}_k^\mathrm{H}\boldsymbol{\Gamma}_i\boldsymbol{T}_i\boldsymbol{T}_i^\mathrm{H}\boldsymbol{\Gamma}_i^\mathrm{H}\boldsymbol{g}_k - \frac{\xi_k}{M^2}\left|\boldsymbol{g}_k^\mathrm{H}\boldsymbol{\Gamma}_k\boldsymbol{T}_k\mathbf{1}\right|^2$$

and $x_k(\boldsymbol{\xi}) = \|\boldsymbol{g}_k\|_2^2 + y_k(\boldsymbol{\xi})$, the minimum MSE reads as

$$\overline{\mathrm{MMSE}}_{k,\mathrm{scalar}}^{\mathrm{MAC}} = \frac{1}{\xi_k}\frac{1}{\frac{1}{\xi_k} + \frac{1}{M^2}\frac{1}{x_k(\boldsymbol{\xi})}\left|\boldsymbol{g}_k^\mathrm{H}\boldsymbol{\Gamma}_k\boldsymbol{T}_k\mathbf{1}\right|^2}. \tag{13}$$

For diagonal $\boldsymbol{D}$, $\boldsymbol{a}^\mathrm{H}\boldsymbol{D}^2\boldsymbol{a} - \frac{1}{M}|\boldsymbol{a}^\mathrm{H}\boldsymbol{D}\mathbf{1}|^2 = \boldsymbol{a}^\mathrm{H}\boldsymbol{D}\boldsymbol{\Pi}\boldsymbol{D}\boldsymbol{a} > 0$ with the projector $\boldsymbol{\Pi} = \mathbf{I}_M - \frac{1}{M}\mathbf{1}\mathbf{1}^\mathrm{T}$. Thus, $x_k(\boldsymbol{\xi}) > 0$. The QoS power allocation problem can be written as [cf. (8)]

$$\min_{\boldsymbol{\xi}\geq\mathbf{0}} \mathbf{1}^\mathrm{T}\boldsymbol{\xi} \quad \text{s.t.:} \quad \forall k\text{: } \varepsilon_k^{-1}J_k(\boldsymbol{\xi}) \leq \xi_k \tag{14}$$

with the interference of user k

$$J_k(\boldsymbol{\xi}) = \left(\frac{1}{\xi_k} + \frac{1}{M^2}\frac{1}{x_k(\boldsymbol{\xi})}\left|\boldsymbol{g}_k^\mathrm{H}\boldsymbol{\Gamma}_k\boldsymbol{T}_k\mathbf{1}\right|^2\right)^{-1} \tag{15}$$

Algorithm 1 Power Minimization

1: Initialize: $l \leftarrow 0$, random initialization for $\boldsymbol{p}_k^{(0)}$
2: **repeat**
3: $l \leftarrow l+1$, execute commands for all $k \in \{1, \ldots, K\}$ and for all $m \in \{1, \ldots, M\}$
4: $\boldsymbol{f}_k^{(l,m)} \leftarrow$ update BC receiver using (7)
5: $\boldsymbol{t}_k^{(l,m)} \leftarrow$ BC-to-MAC conversion (see Section II-A)
6: $\xi_k^{(l+1)} \leftarrow \frac{1}{\varepsilon_k}J_k(\boldsymbol{\xi}^{(l)})$
7: $\boldsymbol{t}_k^{(l+1,m)} \leftarrow \boldsymbol{\tau}_k^{(l,m)}\sqrt{\xi_k^{(l+1)}}$
8: $\boldsymbol{g}_k^{(l+1)} \leftarrow$ update MAC receiver using (9)
9: $\boldsymbol{p}_k^{(l+1)} \leftarrow$ MAC to BC conversion (see Section II-A)
10: **until** $\left|\boldsymbol{\xi}^{(l+1)} - \boldsymbol{\xi}^{(l)}\right| \leq \delta$

and where $\overline{\mathrm{MMSE}}_{k,\mathrm{scalar}}^{\mathrm{MAC}} = J_k(\boldsymbol{\xi})/\xi_k$. Collecting the interferences in $\boldsymbol{J}(\boldsymbol{\xi}) = [J_1(\boldsymbol{\xi}), \ldots, J_K(\boldsymbol{\xi})]^\mathrm{T}$ leads to a standard interference function [10]. Positivity of $\boldsymbol{J}(\boldsymbol{\xi})$ follows from $\boldsymbol{\xi} \geq \mathbf{0}$ and $x_k(\boldsymbol{\xi}) > 0$. Monotonicity can be seen from the property of x_k to be monotonically increasing in $\boldsymbol{\xi}$. Finally, we have $zx_k(\boldsymbol{\xi}) > x_k(z\boldsymbol{\xi})$ for $z > 1$ and, $zJ_k(\boldsymbol{\xi}) > J_k(z\boldsymbol{\xi})$ thus demonstrating scalability. These properties imply that $\boldsymbol{J}(\boldsymbol{\xi})$ is a standard interference function and, if (14) is feasible, the fixed point iteration $\boldsymbol{\xi}^{(\ell)} = \boldsymbol{E}^{-1}\boldsymbol{J}(\boldsymbol{\xi}^{(\ell-1)})$ with $\boldsymbol{E} = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_K)$ converges to the global optimum of (14) as proven in [10].

V. Algorithmic Solution

The pseudocode given in Algorithm 1 finds a feasible solution of (6), if any. In every loop, the BC receivers are updated in line 4. After the BC-to-MAC transform, the MAC power allocation is recomputed based on the interference function $\boldsymbol{J}(\boldsymbol{\xi})$ [see (15)] in line 6 and the MAC receivers are updated in line 8. Due to the MAC-to-BC transform in line 9, this translates into an update of the BC precoders. Note that every step of Algorithm 1 either reduces the average MSEs or the average transmit power. Due to the existence of an infimum for feasible (6) [see Section VI], this property implies that the power converges. We also observe that the filters converge.

VI. QoS Feasibility

In Theorem 1, we present a feasibility test for given average MMSE requirements applicable for imperfect CSI. A similar result was shown in [17] for the vector BC but assuming perfect CSI. For the optimal equalizers [see (9)], the average MAC MMSE achieved for user k can be expressed as

$$\varepsilon_k^{\mathrm{MAC}} = 1 - \boldsymbol{\mu}_k^\mathrm{H}\left(\sum_{i=1}^K \boldsymbol{R}_i + \sigma^2\mathbf{I}_N\right)^{-1}\boldsymbol{\mu}_k \tag{16}$$

where $\boldsymbol{R}_k = \mathrm{E}[\boldsymbol{H}_k\boldsymbol{C}_{\eta_k}^{-\mathrm{H}/2}\boldsymbol{t}_k(\boldsymbol{H}_k)\boldsymbol{t}_k^\mathrm{H}(\boldsymbol{H}_k)\boldsymbol{C}_{\eta_k}^{-1/2}\boldsymbol{H}_k^\mathrm{H}|v]$. With $\boldsymbol{C}_k = \mathrm{E}[(\boldsymbol{H}_k\boldsymbol{C}_{\eta_k}^{-\mathrm{H}/2}\boldsymbol{t}_k(\boldsymbol{H}_k) - \boldsymbol{\mu}_k)(\boldsymbol{t}_k^\mathrm{H}(\boldsymbol{H}_k)\boldsymbol{C}_{\eta_k}^{-1/2}\boldsymbol{H}_k^\mathrm{H} - \boldsymbol{\mu}_k^\mathrm{H})|v]$, we have $\boldsymbol{R}_k = \boldsymbol{C}_k + \boldsymbol{\mu}_k\boldsymbol{\mu}_k^\mathrm{H}$. Note that we have introduced a non-unit variance σ^2 of the noise $\boldsymbol{\eta}$ in the dual MAC. Introducing $\boldsymbol{\Theta} = \boldsymbol{H}\boldsymbol{T}$ where $\boldsymbol{H} = [\boldsymbol{H}_1\boldsymbol{C}_{\eta_1}^{-\mathrm{H}/2}, \ldots, \boldsymbol{H}_K\boldsymbol{C}_{\eta_K}^{-\mathrm{H}/2}]$ and $\boldsymbol{T} = \mathrm{blockdiag}(\boldsymbol{t}_1(\boldsymbol{H}_1), \ldots, \boldsymbol{t}_K(\boldsymbol{H}_K))$ yields

$$\varepsilon_k^{\mathrm{MAC}} = 1 - \left[\mathrm{E}[\boldsymbol{\Theta}^\mathrm{H}|v]\left(\mathrm{E}[\boldsymbol{\Theta}\boldsymbol{\Theta}^\mathrm{H}|v] + \sigma^2\mathbf{I}_N\right)^{-1}\mathrm{E}[\boldsymbol{\Theta}|v]\right]_{k,k}.$$

Due to the imperfect CSI at the transmitter, the average MMSEs cannot be arbitrarily reduced in a simultaneous way, even when $K \geq N$. This observation is in contrast to the perfect CSI case (see [17]). Employing the above expression for $\varepsilon_k^{\mathrm{MAC}}$, we get for the average sum MMSE $\varepsilon^{\mathrm{MAC}} = \sum_{k=1}^K \varepsilon_k^{\mathrm{MAC}}$

$$\varepsilon^{\mathrm{MAC}} = K - \operatorname{tr}\left\{ \mathrm{E}[\boldsymbol{\Theta}^{\mathrm{H}}|v] \left(\mathrm{E}[\boldsymbol{\Theta}\boldsymbol{\Theta}^{\mathrm{H}}|v] + \sigma^2 \mathbf{I}_N\right)^{-1} \mathrm{E}[\boldsymbol{\Theta}|v] \right\}.$$

By setting the noise variance to zero, i.e., $\sigma^2 = 0$, we observe that the average MMSEs collected in $\boldsymbol{\varepsilon} = [\varepsilon_1^{\mathrm{MAC}}, \ldots, \varepsilon_K^{\mathrm{MAC}}]^{\mathrm{T}}$ for any finite total average power allocation satisfies

$$\mathbf{1}^{\mathrm{T}}\boldsymbol{\varepsilon} > K - \operatorname{tr}\{\boldsymbol{X}\} \tag{17}$$

with $\boldsymbol{X} = \mathrm{E}[\boldsymbol{T}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}|v](\mathrm{E}[\boldsymbol{H}\boldsymbol{T}\boldsymbol{T}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}|v])^{-1}\,\mathrm{E}[\boldsymbol{H}\boldsymbol{T}|v]$. Here, we assume that $\mathrm{E}[\boldsymbol{H}\boldsymbol{T}\boldsymbol{T}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}|v]$ is invertible, which is fulfilled when at least one of the error covariance matrices $\boldsymbol{C}_k$ is non-singular, for example. Equality in (17) is asymptotically achieved when the powers for all users reach infinity.

So far, we have found a necessary condition for the feasibility of QoS targets, i.e., any power allocation with finite sum power achieves an MMSE tuple $\boldsymbol{\varepsilon}$ inside the polytope

$$\mathcal{P} = \left\{ \boldsymbol{\varepsilon} \,|\, \mathbf{1}^{\mathrm{T}}\boldsymbol{\varepsilon} \geq K - \operatorname{tr}\{\boldsymbol{X}\} \text{ and } \forall k:\ 0 \leq \varepsilon_k \leq 1 \right\}. \tag{18}$$

To show that $\mathcal{P}$ is the feasible set of (14), we must prove the converse, i.e., that there exists a power allocation for any tuple inside the polytope $\mathcal{P}$. The mapping from $\boldsymbol{\varepsilon}$ to the power allocation resuls from equating $\varepsilon_k^{\mathrm{MAC}}$ [see (16)] with the target $\varepsilon_k^{\mathrm{target}}$. The resulting fixed point is unique due to the properties of interference functions [10]. Then, if the fixed point exists, the aforementioned mapping is bijective. In [15], Kennan established sufficient conditions for the existence of the fixed point $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{\varepsilon}^{\mathrm{target}})$, viz.,

$$\boldsymbol{f}(\mathbf{0}; \boldsymbol{\varepsilon}^{\mathrm{target}}) \geq \mathbf{0} \tag{19}$$

$$\exists\, \boldsymbol{a} > \mathbf{0} \quad \text{with} \quad \boldsymbol{f}(\boldsymbol{a}; \boldsymbol{\varepsilon}^{\mathrm{target}}) > \boldsymbol{a} \tag{20}$$

$$\exists\, \boldsymbol{b} > \boldsymbol{a} \quad \text{with} \quad \boldsymbol{f}(\boldsymbol{b}; \boldsymbol{\varepsilon}^{\mathrm{target}}) < \boldsymbol{b}. \tag{21}$$

For finding $\boldsymbol{f}$, apply the matrix inversion lemma to (16) to get

$$\varepsilon_k^{\mathrm{MAC}} = \left(1 + \xi_k \boldsymbol{\varphi}_k^{\mathrm{H}} \boldsymbol{A}_k^{-1} \boldsymbol{\varphi}_k\right)^{-1} \tag{22}$$

with $\boldsymbol{\mu}_k = \sqrt{\xi_k}\boldsymbol{\varphi}_k$, $\boldsymbol{A}_k = \sum_{i=1}^K \xi_i \boldsymbol{\Phi}_i + \sum_{j\neq k} \xi_j \boldsymbol{\varphi}_j \boldsymbol{\varphi}_j^{\mathrm{H}} + \sigma^2 \mathbf{I}_N$ and $\boldsymbol{C}_k = \xi_k \boldsymbol{\Phi}_k$. Equating $\varepsilon_k^{\mathrm{MAC}} = \varepsilon_k^{\mathrm{target}} > 0$ shows that the optimal powers are the fixed points $f_k(\boldsymbol{\xi}; \boldsymbol{\varepsilon}^{\mathrm{target}}) = \xi_k$ with

$$f_k(\boldsymbol{\xi}; \boldsymbol{\varepsilon}^{\mathrm{target}}) := \left(\left(\varepsilon_k^{\mathrm{target}}\right)^{-1} - 1\right)\left(\boldsymbol{\varphi}_k^{\mathrm{H}} \boldsymbol{A}_k^{-1} \boldsymbol{\varphi}_k\right)^{-1}. \tag{23}$$

Clearly, $f_k(\boldsymbol{\xi}; \boldsymbol{\varepsilon}^{\mathrm{target}})$ fulfills the first requirement (19) due to

$$f_k(\mathbf{0}; \boldsymbol{\varepsilon}^{\mathrm{target}}) = \frac{1 - \varepsilon_k^{\mathrm{target}}}{\varepsilon_k^{\mathrm{target}}} \frac{\sigma^2}{\|\boldsymbol{\varphi}_k\|_2^2} \geq 0 \quad \text{if} \quad 0 < \varepsilon_k^{\mathrm{target}} \leq 1.$$

Consequently, we have found a lower bound for $f_k(\boldsymbol{\xi}; \boldsymbol{\varepsilon}^{\mathrm{target}})$, that is, $f_k(\boldsymbol{\xi}; \boldsymbol{\varepsilon}^{\mathrm{target}}) \geq \frac{1-\varepsilon_k^{\mathrm{target}}}{\varepsilon_k^{\mathrm{target}}} \frac{\sigma^2}{\|\boldsymbol{\varphi}_k\|_2^2}$ for $\boldsymbol{\xi} \geq \mathbf{0}$. Obviously, a lower bound to $f_k(\boldsymbol{\xi}; \boldsymbol{\varepsilon}^{\mathrm{target}})$ for all k can be found by

$$\overline{a} = \min_k \frac{1 - \varepsilon_k^{\mathrm{target}}}{\varepsilon_k^{\mathrm{target}}} \frac{\sigma^2}{\|\boldsymbol{\varphi}_k\|_2^2} \quad \text{for} \quad \varepsilon_k^{\mathrm{target}} < 1. \tag{24}$$

Bearing in mind that $\overline{a} > 0$ for $\varepsilon_k^{\mathrm{target}} < 1$ and choosing $a < \overline{a}$, the second condition (20) is satisfied using $\boldsymbol{a} = a\mathbf{1}$. To demonstrate that the third requirement (21) is also accomplished, we need to consider the following two cases.

1) Number of transmit antennas greater than or equal to the number of users ($N \geq K$): first, we apply the matrix inversion lemma to the denominator in (23) to obtain

$$\boldsymbol{\varphi}_k^{\mathrm{H}} \boldsymbol{\Phi}^{-1} \left[\mathbf{I} - \boldsymbol{B}_{\bar{k}} \left(\boldsymbol{\Xi}_{\bar{k}}^{-1} + \boldsymbol{B}_{\bar{k}}^{\mathrm{H}} \boldsymbol{\Phi}^{-1} \boldsymbol{B}_{\bar{k}}\right)^{-1} \boldsymbol{B}_{\bar{k}}^{\mathrm{H}} \boldsymbol{\Phi}^{-1}\right] \boldsymbol{\varphi}_k$$

where $\sum_{j\neq k} \xi_j \boldsymbol{\varphi}_j \boldsymbol{\varphi}_j^{\mathrm{H}} = \boldsymbol{B}_{\bar{k}} \boldsymbol{\Xi}_{\bar{k}} \boldsymbol{B}_{\bar{k}}^{\mathrm{H}}$ with $\boldsymbol{\Xi}_{\bar{k}} = \operatorname{diag}(\xi_j)_{j\neq k}$, $\boldsymbol{B}_{\bar{k}} = [\boldsymbol{\varphi}_j]_{j\neq k}$, and $\boldsymbol{\Phi} = \sum_{i=1}^K \xi_i \boldsymbol{\Phi}_i + \sigma^2 \mathbf{I}_N$. Thus, $f_k(\boldsymbol{\xi}; \boldsymbol{\varepsilon}^{\mathrm{target}})$ can be upper bounded by lower bounding its denominator via [see (23)]

$$\boldsymbol{\varphi}_k^{\mathrm{H}} \boldsymbol{A}_k^{-1} \boldsymbol{\varphi}_k \geq \boldsymbol{\psi}_k^{\mathrm{H}} \left(\mathbf{I} - \boldsymbol{D}_{\bar{k}} \left(\boldsymbol{D}_{\bar{k}}^{\mathrm{H}} \boldsymbol{D}_{\bar{k}}\right)^{-1} \boldsymbol{D}_{\bar{k}}^{\mathrm{H}}\right) \boldsymbol{\psi}_k \tag{25}$$

with $\boldsymbol{\psi}_k = \boldsymbol{\Phi}^{-1/2}\boldsymbol{\varphi}_k$ and $\boldsymbol{D}_{\bar{k}} = \boldsymbol{\Phi}^{-1/2}\boldsymbol{B}_{\bar{k}}$. The equality in (25) holds for $\forall k\colon \xi_k \to \infty$. We also introduce the orthogonal projector $\boldsymbol{\Pi}_k$ onto the complement of the range space of $\boldsymbol{D}_{\bar{k}}$. Hence, the righthand side of (25) can be rewritten as $\boldsymbol{\psi}_k^{\mathrm{H}} \boldsymbol{\Pi}_k \boldsymbol{\psi}_k$. The third condition (21) can be fulfilled by choosing $\boldsymbol{b}$ such that $b_k > (\frac{1}{\varepsilon_k^{\mathrm{target}}} - 1)/(\boldsymbol{\psi}_k^{\mathrm{H}} \boldsymbol{\Pi}_k \boldsymbol{\psi}_k)$, completing the proof that $\mathcal{P}$ is the feasible set of (14) for $N \geq K$.

2) Number of transmit antennas smaller than the number of users ($N < K$): Set $\boldsymbol{b} = \alpha \boldsymbol{b}_0$, where $\boldsymbol{b}_0$ belongs to the simplex $\mathcal{S} = \{\boldsymbol{x} | \sum_k x_k = 1 \text{ and } \forall k\colon x_k \geq 0\}$. For $\alpha \to \infty$ (or $\sigma^2 \to 0$) and $\boldsymbol{b}_0 > \mathbf{0}$, we can rewrite (23) as

$$f_k^{\infty}(\boldsymbol{b}_0; \boldsymbol{\varepsilon}^{\mathrm{target}}) := \frac{\frac{1}{\varepsilon_k^{\mathrm{target}}} - 1}{\boldsymbol{\varphi}_k^{\mathrm{H}} \Big(\sum\limits_i b_{0,i} \boldsymbol{\Phi}_i + \sum\limits_{j\neq k} b_{0,j} \boldsymbol{\varphi}_j \boldsymbol{\varphi}_j^{\mathrm{H}}\Big)^{-1} \boldsymbol{\varphi}_k}$$

where $b_k = \alpha f_k^{\infty}(\boldsymbol{b}_0; \boldsymbol{\varepsilon}^{\mathrm{target}})$. Since $b_{0,k} = f_k^{\infty}(\boldsymbol{b}_0; \boldsymbol{\varepsilon}^{\mathrm{target}})$, we observe that $\boldsymbol{b}_0 - \boldsymbol{f}^{\infty}(\boldsymbol{b}_0; \boldsymbol{\varepsilon}^{\mathrm{target}}) = \mathbf{0}$. The tuple $\boldsymbol{\varepsilon}^{\mathrm{target}}$ has to satisfy $\mathbf{1}^{\mathrm{T}}\boldsymbol{\varepsilon}^{\mathrm{target}} = K - \operatorname{tr}\{\boldsymbol{X}\}$ for $\alpha \to \infty$ or $\sigma^2 \to 0$ [cf. (17)]. In other words, a power allocation $\boldsymbol{b} = \alpha \boldsymbol{b}_0$ with $\boldsymbol{b}_0 > \mathbf{0}$ and $\alpha \to \infty$ achieves a tuple $\boldsymbol{\varepsilon}^{\mathrm{target}}$ that lies in the region $\mathcal{B} = \{\boldsymbol{\varepsilon} | \mathbf{1}^{\mathrm{T}}\boldsymbol{\varepsilon} = K - \operatorname{tr}\{\boldsymbol{X}\} \text{ and } \forall k\colon 0 \leq \varepsilon_k \leq 1\}$ which separates feasible from infeasible targets. Obviously, $\mathcal{B} \subset \mathcal{P}$ [see (18)]. Then, to complete the proof, we need to show that there exists a unique power allocation $\boldsymbol{b} = \alpha \boldsymbol{b}_0$ for any MMSE tuple $\boldsymbol{\varepsilon} \in \mathcal{B}$ with $\boldsymbol{b}_0 \in \mathcal{S}$ and $\alpha \to \infty$.

Define the kth user's SINR as $\frac{1}{\varepsilon_k^{\mathrm{MAC}}} - 1$. When $\alpha \to \infty$, the SINR reduces to the signal-to-interference ratio [cf. (22)] $\mathrm{SIR}_k = b_{0,k} \boldsymbol{\varphi}_k^{\mathrm{H}} (\sum_{i=1}^K b_{0,i} \boldsymbol{\Phi}_i + \sum_{j\neq k} b_{0,j} \boldsymbol{\varphi}_j \boldsymbol{\varphi}_j^{\mathrm{H}})^{-1} \boldsymbol{\varphi}_k$. Now, we write the SIR_k as the power factor $b_{0,k}$ over the function $\mathcal{Q}_k(\boldsymbol{b}_0)$. Fortunately, the functions $\mathcal{Q}_k(\boldsymbol{b}_0)$ satisfy the properties given in [16]. Thus, the existence and uniqueness of the optimal power allocation for the SIR balancing problem

$$\max_{r, \boldsymbol{b}_0}\ r \quad \text{s.t.} \quad \frac{b_{0,k}}{\mathcal{Q}_k(\boldsymbol{b}_0)} = r\, \mathrm{SIR}_{0,k} \quad \forall k \in \{1, \ldots, K\} \tag{26}$$

is guaranteed. Note that we have that $\varepsilon_k^{\mathrm{MAC}} = \frac{1}{1+\mathrm{SIR}_k}$ in the high power regime with $\alpha \to \infty$. Therefore, it holds that $\mathbf{1}^{\mathrm{T}}\boldsymbol{\varepsilon} = \sum_{k=1}^K \frac{1}{1+r\,\mathrm{SIR}_{0,k}}$ is the optimum of (26). Remember from the above discussion that $\boldsymbol{\varepsilon} \in \mathcal{B}$, i.e., $\mathbf{1}^{\mathrm{T}}\boldsymbol{\varepsilon} = K - \operatorname{tr}\{\boldsymbol{X}\}$ for $\alpha \to \infty$. Therefore, the balancing level r can be found via

$$\sum_{k=1}^K \frac{1}{1 + r\,\mathrm{SIR}_{0,k}} = K - \operatorname{tr}\{\boldsymbol{X}\}.$$

Note that this equation only has a single solution $r > 0$ since $\frac{1}{1+r\,\mathrm{SIR}_{0,k}}$ is monotonically decreasing for $r > 0$ and all k. In

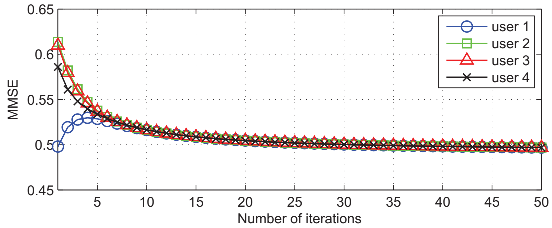

Fig. 2: Example of algorithm execution.

particular, $r = 1$ for SIR targets resulting from MMSE targets out of $\mathcal{B}$, i.e., $\sum_{k=1}^{K} \frac{1}{1+\text{SIR}_{0,k}} = K - \text{tr}\{\boldsymbol{X}\}$ is fulfilled by the SIR targets. Summing up, we have shown based on results for SIR balancing that a unique power allocation $\boldsymbol{b} = \alpha \boldsymbol{b}_0$ with $\boldsymbol{b}_0 \in \mathcal{S}$ and $\alpha \to \infty$ always exists for any MMSE targets $\boldsymbol{\varepsilon}' \in \mathcal{B}$ such that $\boldsymbol{f}(\boldsymbol{b}; \boldsymbol{\varepsilon}') = \boldsymbol{b}$ where the entries of $\boldsymbol{f}$ are defined in (23). Since $\boldsymbol{f}(\boldsymbol{b}; \boldsymbol{\varepsilon}^{\text{target}})$ is decreasing in $\boldsymbol{\varepsilon}^{\text{target}}$ [see (23)], we can infer for any $\boldsymbol{\varepsilon}^{\text{target}} = \beta \boldsymbol{\varepsilon}'$ with $\beta > 1$ that

$$\boldsymbol{f}(\boldsymbol{b}; \boldsymbol{\varepsilon}^{\text{target}}) < \boldsymbol{b}.$$

Hence, the third requirement (21) is also fulfilled for $N < K$. This completes the proof of the following theorem.

Theorem 1. *The QoS problem*

$$\min_{\boldsymbol{\xi} \geq \boldsymbol{0}, \{\boldsymbol{g}_k\}_{k=1}^{K}} \mathbf{1}^{\mathrm{T}} \boldsymbol{\xi} \quad \textit{s.t.} \quad \overline{\mathit{MSE}}_k^{\mathit{MAC}} \leq \varepsilon_k$$

has a solution, i.e., the targets $\boldsymbol{\varepsilon}^{\text{target}} = [\varepsilon_1, \ldots, \varepsilon_K]^{\mathrm{T}}$ are feasible, if and only if $\boldsymbol{\varepsilon}^{\text{target}} \in \mathcal{P}$ where $\mathcal{P}$ is defined in (18).

Remarks: Theorem 1 is a generalization of Theorem III.1 in [17]. For error-free CSI, i.e., $\forall k$: $\boldsymbol{C}_{\boldsymbol{H}_k} = \mathrm{E}[\boldsymbol{h}_k \boldsymbol{h}_k^{\mathrm{H}} | v] = \boldsymbol{0}$ with $\boldsymbol{h}_k = \text{vec}(\boldsymbol{H}_k)$, we have $\text{tr}\{\boldsymbol{X}\} = N$ and the bound reduces to $\mathbf{1}^{\mathrm{T}} \boldsymbol{\varepsilon} > K - N$. Our observation is that $\text{tr}\{\boldsymbol{X}\} = R$ for $\forall k$: $\boldsymbol{C}_{\boldsymbol{H}_k} = \sigma_{h_k}^2 \mathbf{I}_{RN}$ where R is the number of antennas per user. If this observation is true in general, we can infer that the feasible region for single-antenna receivers equals that of SISO systems if $\forall k$: $\boldsymbol{C}_{\boldsymbol{H}_k} = \sigma_{h_k}^2 \mathbf{I}_{RN}$. Contrary to the perfect CSI case, $\text{tr}\{\boldsymbol{X}\}$ need not be integer for $\boldsymbol{C}_{\boldsymbol{H}_k} \neq \sigma_{h_k}^2 \mathbf{I}_{RN}$.

VII. Simulation Results

We present the results of a simulation for $N = 4$ transmit antennas, $K = 4$ users, and $R = 2$ receive antennas per user. Fig. 2 shows the average MMSEs vs. the number of iterations. The result is the mean of $M = 4000$ channel realizations and the average MMSE targets are $\varepsilon_k = 0.45$, $\forall k$, that is, $\mathrm{E}[R_k] \geq -\log_2(0.45) = 1.152$, $\forall k$. As we can see, these rate targets are not reached even when we let the power grow without restriction due to QoS constraints infeasibility. This result agrees with Theorem 1 stating that the sum of average MSEs is lower bounded by $K - \text{tr}\{\boldsymbol{X}\}$. Using the optimal MAC precoders from Algorithm 1, we find that $\text{tr}\{\boldsymbol{X}\} = 2$ for partial CSI v translated into Rayleigh channels with $\forall k$: $\boldsymbol{C}_{\boldsymbol{H}_k} = \sigma_{h_k}^2 \mathbf{I}_{RN}$. The resulting average sum MSE lower bound is $\mathbf{1}^{\mathrm{T}} \boldsymbol{\varepsilon} \geq 2$, i.e., $\varepsilon_k \geq 0.5, \forall k$ if the average MSEs are balanced.

VIII. Conclusion

We proposed an algorithm for rate balancing via MMSE balancing in MU-MIMO systems. At each iteration, using the average MSE duality between the MAC and the BC, the precoders and the receivers are updated, and the transmit power is minimized by means of standard interference functions. We also described the MMSE feasibility region. If we choose QoS constraints inside $\mathcal{P}$, convergence and optimality are ensured due to [10].

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