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Block Toeplitz Matrices: Asymptotic Results and Applications

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Abstract

The present monograph studies the asymptotic behaviour of eigenvalues, products and functions of block Toeplitz matrices generated by the Fourier coefficients of a continuous matrix-valued function. This study is based on the concept of asymptotically equivalent sequences of non-square matrices. The asymptotic results on block Toeplitz matrices obtained are applied to vector asymptotically wide sense stationary processes. Therefore, this monograph is a generalization to block Toeplitz matrices of the Gray monograph entitled "Toeplitz and circulant matrices: A review", which was published in the second volume of Foundations and Trends in Communications and Information Theory, and which is the simplest and most famous introduction to the asymptotic theory on Toeplitz matrices.

1

Introduction

The Gray monograph entitled “Toeplitz and circulant matrices: A review” [8], which was published in the second volume of Foundations and Trends in Communications and Information Theory, is the simplest and most famous introduction to the asymptotic theory on Toeplitz matrices. The secret of the success of that monograph lies in the simplicity of the mathematical tools that Gray used to prove important results on Toeplitz matrices. Specifically, he proved asymptotic results on eigenvalues, products and inverses of Toeplitz matrices by using mainly the concept of asymptotically equivalent sequences of matrices, which he introduced in [7].

The present monograph is a generalization of the Gray monograph to block Toeplitz matrices, which is a type of matrices frequently used in Communications, Information Theory and Signal Processing, because, for instance, matrix representations of discrete-time causal finite impulse response (FIR) multiple-input multiple-output (MIMO) filters and correlation matrices of vector wide sense stationary (WSS) processes are block Toeplitz. Therefore, the present monograph deals with the asymptotic behaviour of eigenvalues, products and inverses of block Toeplitz matrices, and, as in [8], the concept of asymptotically equivalent sequences of matrices is the key concept in it. However, since the blocks of block Toeplitz matrices are not, in general, square, we extend here the Gray definition of asymptotically equivalent sequences of matrices to sequences of non-square matrices.

Unlike in [8], where powers and inverses of Toeplitz matrices were the only functions of Toeplitz matrices studied, in the present monograph we cover any function of block Toeplitz matrices. Furthermore, while the Toeplitz matrices that Gray considered in [8] were generated by the Fourier coefficients of a function in the Wiener class (which is a special subset of the set of continuous functions) we consider here block Toeplitz matrices generated by the Fourier coefficients of any continuous matrix-valued function. Observe that even for the case of Toeplitz matrices (that is, block Toeplitz matrices with 1×1 blocks) the type of matrices considered here is more general, since they are Toeplitz matrices generated by the Fourier coefficients of continuous functions without imposing the Wiener condition.

Since the continuous functions are Riemann integrable, the integral that we use in the monograph is the Riemann integral. Thus, as in [8], it is not required to use the Lebesgue integral, an integral that is not included in a typical engineering background. We recall here that continuous functions are the most representative type of Riemann integrable functions, since the functions that are Riemann integrable on a closed interval are those which are continuous except on a set of Lebesgue measure zero and which are bounded.

Although there are some new results in the present monograph, most of them were given by the authors in [12] and [13], however, here they are presented in a tutorial manner and proved in detail.

The rest of the monograph is organized as follows. In Section 2 the mathematical preliminaries are given. We review the two matrix norms that are used in Section 3 to define the concept of asymptotically equivalent sequences of matrices: the spectral norm and the Frobenius norm. Furthermore, we review the concept of function of a matrix and we give some examples of functions of matrices, such as, the powers, the exponential or the inverse of a matrix.

Section 3 is devoted to the concept of asymptotically equivalent sequences of non-square matrices. We give its definition, its basic properties and we study its relation to the concept of function of a matrix.

In Section 4 we review the definition of a block Toeplitz matrix and show that matrix representations of discrete-time causal FIR MIMO filters and correlation matrices of vector WSS processes are block Toeplitz. However, most of the section is devoted to the presentation of several non-asymptotic properties of a sequence of block Toeplitz matrices generated by the Fourier coefficients of a continuous matrix-valued function.

In Section 5 we study a special type of block Toeplitz matrices: block circulant matrices. Moreover, we define the sequence of block circulant matrices generated by a continuous matrix-valued function, and we prove several non-asymptotic properties of this sequence.

Section 6 is devoted to the study of the asymptotic behaviour of block Toeplitz matrices. We begin by proving that the sequence of block Toeplitz matrices and the sequence of block circulant matrices generated by the same continuous matrix-valued function are asymptotically equivalent. Based on this result and the results obtained in the previous sections, we analyse the asymptotic behaviour of eigenvalues, products and functions of block Toeplitz matrices generated by the Fourier coefficients of a continuous matrix-valued function. In particular, we prove the most famous asymptotic result on block Toeplitz matrices: the Szegö theorem for block Toeplitz matrices.[1] This theorem deals with the arithmetic mean of the eigenvalues of functions of large Hermitian block Toeplitz matrices, and it has found different applications in Information Theory and Signal Processing (see, e.g., [6, 12, 13, 15, 19, 23]). As a matter of fact, in Section 6 we prove that the Szegö theorem for block Toeplitz matrices is also true if the sequence of Hermitian block Toeplitz matrices is replaced by any other asymptotically equivalent sequence of Hermitian matrices.

Finally, the theory on block Toeplitz matrices developed in this monograph is applied in Section 7 to study some vector non-stationary

[1] The Szegö theorem for block Toeplitz matrices is the generalization to block Toeplitz matrices of the famous result on Toeplitz matrices given by Szegö in [9, p. 64].

processes. Specifically, we define there the concept of vector asymptotically WSS (AWSS) process, which is based on the concept of asymptotically equivalent sequences of matrices, and we give several interesting properties and examples of this kind of process. Moreover, we compute the differential entropy rate of certain vector AWSS processes.

2

Mathematical Preliminaries

2.1 Matrix Norms

In this monograph $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural numbers (i.e., the set of positive integers), the set of integer numbers, the set of (finite) real numbers, and the set of (finite) complex numbers, respectively. We begin this section by reviewing the concept of matrix norm on $\mathbb{C}^{m\times n}$ (see, e.g., [2]), where $\mathbb{C}^{m\times n}$, with $m, n \in \mathbb{N}$, is the set of all $m \times n$ complex matrices.

Definition 2.1. A *matrix norm* on $\mathbb{C}^{m\times n}$ is a function $\|\cdot\|: \mathbb{C}^{m\times n} \to [0, \infty)$ that satisfies the three following properties:

(1) $\|A\| = 0$ if and only if $A = 0_{m\times n}$, where $0_{m\times n}$ is the $m \times n$ zero matrix.
(2) $\|\alpha A\| = |\alpha|\|A\|$ for all $\alpha \in \mathbb{C}$ and $A \in \mathbb{C}^{m\times n}$.
(3) $\|A + B\| \leq \|A\| + \|B\|$ for all $A, B \in \mathbb{C}^{m\times n}$.

A matrix norm $\|\cdot\|$ on $\mathbb{C}^{m\times n}$ is *unitarily invariant* if $\|UAV\| = \|A\|$ for all $A \in \mathbb{C}^{m\times n}$ and for all unitary[1] matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$.

[1] An $n \times n$ invertible matrix U is unitary if $U^{-1} = U^*$, where $U^* = \left(\overline{U}\right)^\top$ with $\top$ denoting transpose (that is, $*$ denotes conjugate transpose).

Property (3) of Definition 2.1 is called triangle inequality.

We now review the definition of two well-known unitarily invariant matrix norms that we will need for the definition of asymptotically equivalent sequences of matrices in Section 3: the spectral norm and the Frobenius norm. Moreover, in Propositions 2.1, 2.2 and 2.3 we review the properties of these two matrix norms that will be used in the monograph. All those properties can be found, e.g., in [2].

2.1.1 The Spectral Norm

Definition 2.2. *The spectral norm* of $A \in \mathbb{C}^{m\times n}$ is defined as

$$\|A\|_2 := \max_{\substack{\mathbf{x} \in \mathbb{C}^{n\times 1} \\ \mathbf{x} \neq 0_{n\times 1}}} \left(\frac{\mathbf{x}^* A^* A \mathbf{x}}{\mathbf{x}^*\mathbf{x}}\right)^{\frac{1}{2}}.$$

Thus, for instance,

$$\|I_n\|_2 = \max_{\substack{\mathbf{x} \in \mathbb{C}^{n\times 1} \\ \mathbf{x} \neq 0_{n\times 1}}} \left(\frac{\mathbf{x}^*\mathbf{x}}{\mathbf{x}^*\mathbf{x}}\right)^{\frac{1}{2}} = \max_{\substack{\mathbf{x} \in \mathbb{C}^{n\times 1} \\ \mathbf{x} \neq 0_{n\times 1}}} 1 = 1, \quad \forall n \in \mathbb{N},$$

where I_n denotes the $n \times n$ identity matrix.

The following proposition provides some useful properties of the spectral norm. In particular, in that proposition the spectral norm is written in terms of singular values and eigenvalues. Here and subsequently, $\sigma_j(A)$, $1 \leq j \leq \min\{m,n\}$, are the singular values of $A \in \mathbb{C}^{m\times n}$ numbered in a non-increasing order, and $\lambda_k(B)$, $1 \leq k \leq n$, denote the eigenvalues of a diagonalizable matrix $B \in \mathbb{C}^{n\times n}$.

Proposition 2.1. The spectral norm is a unitarily invariant matrix norm, and it satisfies

$$\|A\|_2 = \left\|\overline{A}\right\|_2 = \left\|A^\top\right\|_2 = \left(\max_{1\leq k\leq n} \lambda_k(A^*A)\right)^{\frac{1}{2}} = \sigma_1(A) \tag{2.1}$$

and

$$\|AB\|_2 \leq \|A\|_2\|B\|_2 \tag{2.2}$$

for all $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. If A is Hermitian then

$$\|A\|_2 = \max_{1\le k\le n} |\lambda_k(A)|.$$

Observe that if we consider an $n \times n$ diagonal matrix

$$A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n) = \operatorname{diag}_{1\le k\le n}(\alpha_k) := \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n \end{pmatrix},$$

then from (2.1) we obtain

$$\begin{aligned} \|A\|_2 &= \left(\max_{1\le k\le n} \lambda_k \left(\operatorname{diag}\left(|\alpha_1|^2, \ldots, |\alpha_n|^2 \right) \right) \right)^{\frac{1}{2}} \\ &= \left(\max_{1\le k\le n} |\alpha_k|^2 \right)^{\frac{1}{2}} = \max_{1\le k\le n} |\alpha_k|. \end{aligned}$$

2.1.2 The Frobenius Norm

Definition 2.3. *The Frobenius norm* of an $m \times n$ matrix $A = (a_{j,k})_{1\le j\le m, 1\le k\le n}$ is defined as

$$\|A\|_{\mathrm{F}} := \left(\sum_{j=1}^{m} \sum_{k=1}^{n} |a_{j,k}|^2 \right)^{\frac{1}{2}}.$$

We now give some useful properties of the Frobenius norm.

Proposition 2.2. The Frobenius norm is a unitarily invariant matrix norm, and it satisfies

$$\|A\|_{\mathrm{F}} = \left\|\overline{A}\right\|_F = \left\|A^\top\right\|_{\mathrm{F}}$$

for all $A \in \mathbb{C}^{m\times n}$. If A is square then

$$|\operatorname{tr}(A)| \le \sqrt{n}\|A\|_{\mathrm{F}},$$

where tr denotes trace.

The following proposition provides two inequalities that relate the Frobenius norm with the spectral norm.

Proposition 2.3. If $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, then

$$\|A\|_2 \le \|A\|_{\mathrm{F}} \tag{2.3}$$

and

$$\|AB\|_{\mathrm{F}} \le \min\{\|A\|_2\|B\|_{\mathrm{F}}, \|A\|_{\mathrm{F}}\|B\|_2\}. \tag{2.4}$$

From (2.3) and (2.4) we deduce that (2.2) is also true if the spectral norm is replaced by the Frobenius norm. However, we will not need this property in the monograph.

2.2 Functions of Matrices

We now review the two concepts of functions of matrices that we will use in the monograph. We begin with the concept of polynomial function of a square matrix.

Definition 2.4. Let $p(x) = \alpha_q x^q + \cdots + \alpha_1 x + \alpha_0$ be a polynomial with coefficients in $\mathbb{C}$. If $A \in \mathbb{C}^{n\times n}$ then $p(A)$ is defined as the $n \times n$ matrix given by

$$p(A) := \alpha_q A^q + \cdots + \alpha_1 A + \alpha_0 I_n.$$

Therefore, powers of matrices are examples of polynomial functions of matrices.

Secondly, we review the concept of function of a diagonalizable matrix.

Definition 2.5. If $A \in \mathbb{C}^{n\times n}$ is diagonalizable, and g is a complex function on the set of eigenvalues of A, $g(A)$ is defined as the $n \times n$ matrix given by

$$g(A) := U\mathrm{diag}(g(\lambda_1(A)), \ldots, g(\lambda_n(A)))U^{-1} \tag{2.5}$$

where

$$A = U\text{diag}(\lambda_1(A), \dots, \lambda_n(A))U^{-1}$$

is any eigenvalue decomposition of A.

It can be proved (see, e.g., [16]) that Definition 2.5 is well defined (that is, (2.5) is independent of the chosen eigenvalue decomposition of the matrix A) and agrees with Definition 2.4.

An interesting example of function of a matrix is the exponential of a matrix that plays an important role in solving systems of first-order linear differential equations (see, e.g., [4, 17]).

Observe that if $A \in \mathbb{C}^{n\times n}$ is an invertible diagonalizable matrix then $A^{-1} = g(A)$ with $g(x) = \frac{1}{x}$, because

$$\begin{aligned} Ag(A) &= U\text{diag}_{1\le k\le n}(\lambda_k(A))U^{-1}U\text{diag}_{1\le k\le n}\left(\frac{1}{\lambda_k(A)}\right)U^{-1} \\ &= U\text{diag}_{1\le k\le n}(\lambda_k(A))I_n\text{diag}_{1\le k\le n}\left(\frac{1}{\lambda_k(A)}\right)U^{-1} \\ &= U\text{diag}_{1\le k\le n}(\lambda_k(A))\text{diag}_{1\le k\le n}\left(\frac{1}{\lambda_k(A)}\right)U^{-1} \\ &= UI_nU^{-1} = UU^{-1} = I_n. \end{aligned}$$

Thus, the inverse of a matrix is another interesting example of function of a matrix.

We now present a result that gives two simple properties of functions of a diagonalizable matrix.

Lemma 2.4. If g and h are two complex functions on the set of eigenvalues of an $n \times n$ diagonalizable matrix A, then

(1) $\text{tr}(g(A)) = \sum_{k=1}^{n} g(\lambda_k(A))$.
(2) $(\alpha g + \beta h)(A) = \alpha g(A) + \beta h(A)$ for all $\alpha, \beta \in \mathbb{C}$.

Proof. Let $A = U\mathrm{diag}(\lambda_1(A), \ldots, \lambda_n(A))U^{-1}$ be an eigenvalue decomposition of A. Then

$$\begin{aligned}
\mathrm{tr}(g(A)) &= \mathrm{tr}(U\mathrm{diag}_{1\le k\le n}(g(\lambda_k(A)))U^{-1}) \\
&= \mathrm{tr}(U^{-1}U\mathrm{diag}_{1\le k\le n}(g(\lambda_k(A)))) \\
&= \mathrm{tr}(\mathrm{diag}_{1\le k\le n}(g(\lambda_k(A)))) = \sum_{k=1}^{n} g(\lambda_k(A)),
\end{aligned}$$

and

$$\begin{aligned}
&(\alpha g + \beta h)(A) \\
&\quad = U\mathrm{diag}_{1\le k\le n}((\alpha g + \beta h)(\lambda_k(A)))U^{-1} \\
&\quad = U\mathrm{diag}_{1\le k\le n}(\alpha g(\lambda_k(A)) + \beta h(\lambda_k(A)))U^{-1} \\
&\quad = U(\mathrm{diag}_{1\le k\le n}(\alpha g(\lambda_k(A))) + \mathrm{diag}_{1\le k\le n}(\beta h(\lambda_k(A))))U^{-1} \\
&\quad = U(\alpha \mathrm{diag}_{1\le k\le n}(g(\lambda_k(A))) + \beta \mathrm{diag}_{1\le k\le n}(h(\lambda_k(A))))U^{-1} \\
&\quad = \alpha U\mathrm{diag}_{1\le k\le n}(g(\lambda_k(A)))U^{-1} + \beta U\mathrm{diag}_{1\le k\le n}(h(\lambda_k(A)))U^{-1} \\
&\quad = \alpha g(A) + \beta h(A).
\end{aligned}$$

□

We finish Section 2 with a result on the spectral norm and the Frobenius norm of a function of a Hermitian matrix.

Lemma 2.5. If g is a complex function on the set of eigenvalues of an $n \times n$ Hermitian matrix A, then

$$\|g(A)\|_2 = \max_{1\le k\le n} |g(\lambda_k(A))| \tag{2.6}$$

and

$$\|g(A)\|_F = \left(\sum_{k=1}^{n} |g(\lambda_k(A))|^2\right)^{\frac{1}{2}}. \tag{2.7}$$

Proof. As A is Hermitian there exists an eigenvalue decomposition of A

$$A = U\mathrm{diag}(\lambda_1(A), \ldots, \lambda_n(A))U^{-1},$$

where the eigenvector matrix U is unitary. Since

$$g(A) = U\mathrm{diag}(g(\lambda_1(A)), \ldots, g(\lambda_n(A)))U^*,$$

U and U^* are unitary matrices, and the spectral norm and the Frobenius norm are unitarily invariant, we have

$$\|g(A)\|_2 = \|\mathrm{diag}(g(\lambda_1(A)), \ldots, g(\lambda_n(A)))\|_2$$

and

$$\|g(A)\|_{\mathrm{F}} = \|\mathrm{diag}(g(\lambda_1(A)), \ldots, g(\lambda_n(A)))\|_{\mathrm{F}}. \qquad \square$$

3

Asymptotically Equivalent Sequences of Matrices

3.1 Definition and Basic Properties

Based on the two matrix norms reviewed in Section 2 we can now proceed to give the key concept of the monograph: the concept of asymptotically equivalent sequences of matrices. This concept was introduced by Gray for sequences of square matrices in [7] and we extended it to sequences of non-square matrices in [13]. The definition of asymptotically equivalent sequences of matrices given in [13] is the one that we review here.

Definition 3.1. Consider two strictly increasing sequences of natural numbers $\{d_n^{(1)}\}$ and $\{d_n^{(2)}\}$. Let A_n and B_n be $d_n^{(1)} \times d_n^{(2)}$ matrices for all $n \in \mathbb{N}$. We say that the sequences $\{A_n\}$ and $\{B_n\}$ are *asymptotically equivalent*, and write $\{A_n\} \sim \{B_n\}$, if

$$\exists M \in [0, \infty) : \quad \|A_n\|_2, \|B_n\|_2 \leq M, \quad \forall n \in \mathbb{N} \tag{3.1}$$

and

$$\lim_{n\to\infty} \frac{\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}} = 0.$$

The original definition of asymptotically equivalent sequences of matrices given by Gray in [7] can be obtained from Definition 3.1 by taking $\{d_n^{(1)}\} = \{d_n^{(2)}\} = \{n\}$. In Appendix A we introduce a definition different from Definition 3.1 that also extends the Gray concept of asymptotically equivalent sequences of matrices to sequences of non-square matrices.

The next lemma, which was given in [13] without a proof, provides some basic properties of asymptotically equivalent sequences of matrices.

Lemma 3.1. Consider two strictly increasing sequences of natural numbers $\{d_n^{(1)}\}$ and $\{d_n^{(2)}\}$. Let A_n, B_n, C_n and D_n be $d_n^{(1)} \times d_n^{(2)}$ matrices for all $n \in \mathbb{N}$.

(1) If $\{A_n\} \sim \{B_n\}$ then $\{B_n\} \sim \{A_n\}$.
(2) If $\{A_n\} \sim \{B_n\}$ and $\{B_n\} \sim \{C_n\}$, then $\{A_n\} \sim \{C_n\}$.
(3) If $\{A_n\} \sim \{B_n\}$ then $\{\alpha A_n\} \sim \{\alpha B_n\}$ for all $\alpha \in \mathbb{C}$.
(4) If $\{A_n\} \sim \{B_n\}$ and $\{C_n\} \sim \{D_n\}$, then $\{A_n + C_n\} \sim \{B_n + D_n\}$.
(5) If $\{A_n\} \sim \{B_n\}$ then $\{\overline{A_n}\} \sim \{\overline{B_n}\}$.
(6) If $\{A_n\} \sim \{B_n\}$ then $\{A_n^\top\} \sim \{B_n^\top\}$.
(7) If $\{A_n\} \sim \{B_n\}$ then $\{A_n^*\} \sim \{B_n^*\}$.

Proof. (1) This property is true because

$$\lim_{n\to\infty} \frac{\|B_n - A_n\|_{\mathrm{F}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{|-1|\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}} = 0.$$

(2) This property holds since

$$\begin{aligned} 0 \le \frac{\|A_n - C_n\|_{\mathrm{F}}}{\sqrt{n}} &= \frac{\|A_n - B_n + B_n - C_n\|_{\mathrm{F}}}{\sqrt{n}} \\ &\le \frac{\|A_n - B_n\|_{\mathrm{F}} + \|B_n - C_n\|_{\mathrm{F}}}{\sqrt{n}} \\ &= \frac{\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}} + \frac{\|B_n - C_n\|_{\mathrm{F}}}{\sqrt{n}} \\ &\to 0. \end{aligned}$$

(3) If $\|A_n\|_2, \|B_n\|_2 \leq M < \infty$ for all $n \in \mathbb{N}$, then

$$\|\alpha A_n\|_2 = |\alpha| \|A_n\|_2 \leq |\alpha| M < \infty$$

and

$$\|\alpha B_n\|_2 = |\alpha| \|B_n\|_2 \leq |\alpha| M$$

for all $n \in \mathbb{N}$. On the other hand,

$$\begin{aligned}\lim_{n\to\infty} \frac{\|\alpha A_n - \alpha B_n\|_{\mathrm{F}}}{\sqrt{n}} &= \lim_{n\to\infty} \frac{|\alpha| \|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}} \\ &= |\alpha| \lim_{n\to\infty} \frac{\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}} = 0.\end{aligned}$$

(4) If $\|A_n\|_2, \|B_n\|_2 \leq M_1 < \infty$ and $\|C_n\|_2, \|D_n\|_2 \leq M_2 < \infty$ for all $n \in \mathbb{N}$, then

$$\|A_n + C_n\|_2 \leq \|A_n\|_2 + \|C_n\|_2 \leq M_1 + M_2 < \infty$$

and

$$\|B_n + D_n\|_2 \leq \|B_n\|_2 + \|D_n\|_2 \leq M_1 + M_2$$

for all $n \in \mathbb{N}$. On the other hand,

$$\begin{aligned}0 &\leq \frac{\|A_n + C_n - (B_n + D_n)\|_{\mathrm{F}}}{\sqrt{n}} \\ &\leq \frac{\|A_n - B_n\|_{\mathrm{F}} + \|C_n - D_n\|_{\mathrm{F}}}{\sqrt{n}} \\ &= \frac{\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}} + \frac{\|C_n - D_n\|_{\mathrm{F}}}{\sqrt{n}} \to 0.\end{aligned}$$

(5) This property is true because

$$\left\|\overline{A_n}\right\|_2 = \|A_n\|_2 \leq M$$

and

$$\left\|\overline{B_n}\right\|_2 = \|B_n\|_2 \leq M$$

for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{\left\|\overline{A_n} - \overline{B_n}\right\|_{\mathrm{F}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\left\|\overline{A_n - B_n}\right\|_{\mathrm{F}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}} = 0.$$

(6) This property holds since

$$\left\|A_n^\top\right\|_2 = \|A_n\|_2 \leq M$$

and

$$\left\|B_n^\top\right\|_2 = \|B_n\|_2 \leq M$$

for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{\left\|A_n^\top - B_n^\top\right\|_{\mathrm{F}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\left\|(A_n - B_n)^\top\right\|_{\mathrm{F}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}} = 0.$$

(7) This basic property of asymptotically equivalent sequences of matrices is a direct consequence of the two previous properties. □

The following lemma was given in [13] and it provides a fundamental property of asymptotically equivalent sequences of matrices. Lemma 3.2 and property (2) of Lemma 3.1 were first proved by Gray in [8] for the case in which $\{d_n^{(1)}\} = \{d_n^{(2)}\} = \{d_n^{(3)}\} = \{n\}$.

Lemma 3.2. Consider three strictly increasing sequences of natural numbers $\{d_n^{(1)}\}$, $\{d_n^{(2)}\}$ and $\{d_n^{(3)}\}$. Let A_n and B_n be $d_n^{(1)} \times d_n^{(2)}$ matrices for all $n \in \mathbb{N}$. Suppose that C_n and D_n are $d_n^{(2)} \times d_n^{(3)}$ matrices for all $n \in \mathbb{N}$. If $\{A_n\} \sim \{B_n\}$ and $\{C_n\} \sim \{D_n\}$, then $\{A_nC_n\} \sim \{B_nD_n\}$.

Proof. If $\|A_n\|_2, \|B_n\|_2 \leq M_1 < \infty$ and $\|C_n\|_2, \|D_n\|_2 \leq M_2 < \infty$ for all $n \in \mathbb{N}$, then

$$\|A_nC_n\|_2 \leq \|A_n\|_2\|C_n\|_2 \leq M_1M_2 < \infty$$

and

$$\|B_nD_n\|_2 \leq \|B_n\|_2\|D_n\|_2 \leq M_1M_2$$

for all $n \in \mathbb{N}$. On the other hand,

$$\begin{aligned} 0 &\leq \frac{\|A_nC_n - B_nD_n\|_{\mathrm{F}}}{\sqrt{n}} \\ &= \frac{\|A_nC_n - A_nD_n + A_nD_n - B_nD_n\|_{\mathrm{F}}}{\sqrt{n}} \end{aligned}$$

$$\leq \frac{\|A_nC_n - A_nD_n\|_{\mathrm{F}} + \|A_nD_n - B_nD_n\|_{\mathrm{F}}}{\sqrt{n}}$$
$$\leq \frac{\|A_n\|_2\|C_n - D_n\|_{\mathrm{F}} + \|A_n - B_n\|_{\mathrm{F}}\|D_n\|_2}{\sqrt{n}}$$
$$\leq \frac{M_1\|C_n - D_n\|_{\mathrm{F}} + M_2\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}}$$
$$= M_1\frac{\|C_n - D_n\|_{\mathrm{F}}}{\sqrt{n}} + M_2\frac{\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}} \to 0. \qquad \square$$

From the proof of Lemma 3.1, we know that Lemma 3.1 is also true if we do not require (3.1) in Definition 3.1. However, as we will now prove, this does not happen with Lemma 3.2, and that is the reason why (3.1) has been included in Definition 3.1. Observe that if

$$\{A_n\} = \{C_n\} = \{\mathrm{diag}(n+1, 0, \ldots, 0)\},$$

and

$$\{B_n\} = \{D_n\} = \{\mathrm{diag}(n, 0, \ldots, 0)\},$$

with

$$\{d_n^{(1)}\} = \{d_n^{(2)}\} = \{d_n^{(3)}\} = \{n\},$$

then the sequences

$$\{\|A_n\|_2\} = \{\|C_n\|_2\} = \{n+1\}$$

and

$$\{\|B_n\|_2\} = \{\|D_n\|_2\} = \{n\}$$

are not bounded,

$$\lim_{n\to\infty} \frac{\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\|C_n - D_n\|_{\mathrm{F}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0,$$

and

$$\lim_{n\to\infty} \frac{\|A_nC_n - B_nD_n\|_{\mathrm{F}}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\|\mathrm{diag}((n+1)^2 - n^2, 0, \ldots, 0)\|_{\mathrm{F}}}{\sqrt{n}}$$
$$= \lim_{n\to\infty} \frac{(n+1)^2 - n^2}{\sqrt{n}} = \lim_{n\to\infty} \frac{2n+1}{\sqrt{n}} = \infty.$$

We finish this subsection with a lemma that gives three additional properties of asymptotically equivalent sequences of matrices.

Lemma 3.3. Consider two strictly increasing sequences of natural numbers $\{d_n^{(1)}\}$ and $\{d_n^{(2)}\}$. Let A_n be a $d_n^{(1)} \times d_n^{(2)}$ matrix for all $n \in \mathbb{N}$.

(1) $\{A_n\} \sim \{A_n\}$ if and only if $\{\|A_n\|_2\}$ is bounded.
(2) If $\{\|A_n\|_\mathrm{F}\}$ is bounded then $\{A_n\} \sim \left\{0_{d_n^{(1)} \times d_n^{(2)}}\right\}$.
(3) Suppose that $\left\{d_n^{(1)}\right\} = \{d_n^{(2)}\}$ and let B_n be a $d_n^{(1)} \times d_n^{(1)}$ matrix for all $n \in \mathbb{N}$. If $\{A_n\} \sim \{B_n\}$ then

$$\{p(A_n)\} \sim \{p(B_n)\}$$

for any polynomial $p(x)$ with coefficients in $\mathbb{C}$.

Proof. (1) Based on Definition 3.1, if $\{A_n\} \sim \{A_n\}$ then $\{\|A_n\|_2\}$ is bounded. The reciprocal is also true because

$$\lim_{n\to\infty} \frac{\|A_n - A_n\|_\mathrm{F}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\left\|0_{d_n^{(1)} \times d_n^{(2)}}\right\|_\mathrm{F}}{\sqrt{n}} = \lim_{n\to\infty} \frac{0}{\sqrt{n}} = 0.$$

(2) If $\|A_n\|_\mathrm{F} \leq M < \infty$ for all $n \in \mathbb{N}$, then

$$\|A_n\|_2 \leq \|A_n\|_\mathrm{F} \leq M$$

and

$$\left\|0_{d_n^{(1)} \times d_n^{(2)}}\right\|_2 = 0 \leq M$$

for all $n \in \mathbb{N}$, and

$$0 \leq \frac{\left\|A_n - 0_{d_n^{(1)} \times d_n^{(2)}}\right\|_\mathrm{F}}{\sqrt{n}} = \frac{\|A_n\|_\mathrm{F}}{\sqrt{n}} \leq \frac{M}{\sqrt{n}} \to 0.$$

(3) We will proceed by induction on the degree q of the polynomial $p(x)$. We begin with the case $q = 0$, that is, $p(x) = \alpha_0$ with $\alpha_0 \in \mathbb{C}$. Since

$$\left\{\left\|\alpha_0 I_{d_n^{(1)}}\right\|_2\right\} = \left\{|\alpha_0| \left\|I_{d_n^{(1)}}\right\|_2\right\} = \{|\alpha_0| \cdot 1\} = \{|\alpha_0|\}$$

is bounded, from property (1) of the present lemma we obtain

$$\{p(A_n)\} = \left\{\alpha_0 I_{d_n^{(1)}}\right\} \sim \left\{\alpha_0 I_{d_n^{(1)}}\right\} = \{p(B_n)\}.$$

Now suppose that the result is true for $q_0 \geq q \geq 0$, and consider the case $q = q_0 + 1$. In this case, $p(x) = \alpha_{q_0+1}x^{q_0+1} + p_{q_0}(x)$ where $\alpha_{q_0+1} \in \mathbb{C}$ and $p_{q_0}(x)$ is a polynomial of degree at most q_0 with coefficients in $\mathbb{C}$. By the induction hypothesis we have

$$\{p_{q_0}(A_n)\} \sim \{p_{q_0}(B_n)\} \tag{3.2}$$

and

$$\{A_n^{q_0}\} \sim \{B_n^{q_0}\}. \tag{3.3}$$

From (3.3), Lemma 3.2, and the fact that $\{A_n\} \sim \{B_n\}$, we obtain

$$\{A_n^{q_0+1}\} = \{A_n^{q_0} A_n\} \sim \{B_n^{q_0} B_n\} = \{B_n^{q_0+1}\},$$

and consequently, applying property (3) of Lemma 3.1 yields

$$\{\alpha_{q_0+1} A_n^{q_0+1}\} \sim \{\alpha_{q_0+1} B_n^{q_0+1}\}. \tag{3.4}$$

Finally, from (3.2), (3.4), and property (4) of Lemma 3.1, we conclude that

$$\begin{aligned}\{p(A_n)\} &= \{\alpha_{q_0+1} A_n^{q_0+1} + p_{q_0}(A_n)\} \\ &\sim \{\alpha_{q_0+1} B_n^{q_0+1} + p_{q_0}(B_n)\} = \{p(B_n)\}.\end{aligned}$$ □

3.2 Asymptotically Equivalent Sequences of Hermitian Matrices

In this subsection, we give three results on asymptotically equivalent sequences of Hermitian matrices.

Lemma 3.4. Consider a strictly increasing sequence of natural numbers $\{d_n\}$. Let A_n and B_n be $d_n \times d_n$ Hermitian matrices for all $n \in \mathbb{N}$, and suppose that $\{A_n\} \sim \{B_n\}$. Then there exists a closed interval $[a, b] \subset \mathbb{R}$ containing the eigenvalues of all the matrices of these two matrix sequences.

Proof. Since $\{A_n\}$ and $\{B_n\}$ are sequences of Hermitian matrices, from (3.1) there exists $M \in [0,\infty)$ such that

$$\max_{1\leq k\leq d_n} |\lambda_k(A_n)|, \max_{1\leq k\leq d_n} |\lambda_k(B_n)| \leq M, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

From (3.5) and the fact that all the eigenvalues of a Hermitian matrix are real, we conclude that the closed interval $[-M, M]$ contains the eigenvalues of all the matrices of the sequences $\{A_n\}$ and $\{B_n\}$. □

Lemma 3.4 was given in [8] for the case in which $\{d_n\} = \{n\}$.

The next result deals with asymptotically equivalent sequences of Hermitian matrices and functions of matrices.

Theorem 3.5. Let $\{d_n\}$, $\{A_n\}$, $\{B_n\}$ and $[a,b]$ be as in Lemma 3.4. If $\{\frac{d_n}{n}\}$ is bounded then

$$\{g(A_n)\} \sim \{g(B_n)\}, \quad \forall g \in C[a,b],$$

where $C[a,b]$ denotes the set of all continuous complex functions on $[a,b]$.

Proof. Fix $g \in C[a,b]$. Applying (2.6) yields

$$\|g(A_n)\|_2, \|g(B_n)\|_2 \leq \max_{a\leq x\leq b} |g(x)|, \quad \forall n \in \mathbb{N},$$

where the existence of $\max_{a\leq x\leq b} |g(x)|$ is guaranteed by the continuity of the real function $|g(x)|$ on the closed interval $[a,b]$.

Let $\epsilon \in (0,\infty)$. Since g is a continuous complex function on $[a,b]$, from the Stone–Weierstrass theorem (see, e.g., [20, p. 159]), there exists a polynomial $p(x)$ with coefficients in $\mathbb{C}$ such that

$$|g(x) - p(x)| < \epsilon, \quad \forall x \in [a,b]. \tag{3.6}$$

Applying property (3) of Lemma 3.3 gives

$$\{p(A_n)\} \sim \{p(B_n)\},$$

and consequently, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\|p(A_n) - p(B_n)\|_F}{\sqrt{n}} < \epsilon, \quad \forall n \geq n_0. \tag{3.7}$$

From the triangle inequality, property (2) of Lemma 2.4, (2.7), (3.6) and (3.7) we conclude that

$$
\begin{aligned}
&\frac{\|g(A_n) - g(B_n)\|_{\mathrm{F}}}{\sqrt{n}} \\
&\quad = \frac{\|g(A_n) - p(A_n) + p(A_n) - p(B_n) + p(B_n) - g(B_n)\|_{\mathrm{F}}}{\sqrt{n}} \\
&\quad \leq \frac{\|g(A_n) - p(A_n)\|_{\mathrm{F}} + \|p(A_n) - p(B_n)\|_{\mathrm{F}} + \|p(B_n) - g(B_n)\|_{\mathrm{F}}}{\sqrt{n}} \\
&\quad = \frac{\|(g-p)(A_n)\|_{\mathrm{F}}}{\sqrt{n}} + \frac{\|p(A_n) - p(B_n)\|_{\mathrm{F}}}{\sqrt{n}} + \frac{\|(p-g)(B_n)\|_{\mathrm{F}}}{\sqrt{n}} \\
&\quad = \frac{\left(\sum_{k=1}^{d_n} |(g-p)(\lambda_k(A_n))|^2\right)^{\frac{1}{2}}}{\sqrt{n}} + \frac{\|p(A_n) - p(B_n)\|_{\mathrm{F}}}{\sqrt{n}} \\
&\qquad + \frac{\left(\sum_{k=1}^{d_n} |(p-g)(\lambda_k(B_n))|^2\right)^{\frac{1}{2}}}{\sqrt{n}} \\
&\quad < \frac{\left(\sum_{k=1}^{d_n} \epsilon^2\right)^{\frac{1}{2}}}{\sqrt{n}} + \epsilon + \frac{\left(\sum_{k=1}^{d_n} \epsilon^2\right)^{\frac{1}{2}}}{\sqrt{n}} \\
&\quad = \frac{\sqrt{d_n}\epsilon}{\sqrt{n}} + \epsilon + \frac{\sqrt{d_n}\epsilon}{\sqrt{n}} \\
&\quad = \left(2\sqrt{\frac{d_n}{n}} + 1\right)\epsilon \\
&\quad \leq (2\sqrt{M} + 1)\epsilon, \quad \forall n \geq n_0,
\end{aligned}
$$

where $M \in (0, \infty)$ is an upper bound of the bounded sequence $\{\frac{d_n}{n}\}$. □

We finish this section with a result that can be obtained as a corollary of Theorem 3.5.

Theorem 3.6. Let $\{d_n\}$, $\{A_n\}$, $\{B_n\}$ and $[a,b]$ be as in Lemma 3.4. Consider $g \in C[a,b]$ and suppose that $\left\{\frac{d_n}{n}\right\}$ is bounded. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{d_n} (g(\lambda_k(A_n)) - g(\lambda_k(B_n))) = 0.$$

Hence if $\left\{\frac{1}{n}\sum_{k=1}^{d_n} g(\lambda_k(B_n))\right\}$ is convergent then $\left\{\frac{1}{n}\sum_{k=1}^{d_n} g(\lambda_k(A_n))\right\}$ is also convergent and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{d_n} g(\lambda_k(A_n)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{d_n} g(\lambda_k(B_n)).$$

Proof. If $\frac{d_n}{n} \leq M < \infty$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}
0 &\leq \left| \frac{1}{n} \sum_{k=1}^{d_n} (g(\lambda_k(A_n)) - g(\lambda_k(B_n))) \right| = \left| \frac{\operatorname{tr}(g(A_n)) - \operatorname{tr}(g(B_n))}{n} \right| \\
&= \frac{|\operatorname{tr}(g(A_n) - g(B_n))|}{n} \leq \frac{\sqrt{d_n}\|g(A_n) - g(B_n)\|_{\mathrm{F}}}{n} \\
&= \sqrt{\frac{d_n}{n}} \frac{\|g(A_n) - g(B_n)\|_{\mathrm{F}}}{\sqrt{n}} \leq \sqrt{M} \frac{\|g(A_n) - g(B_n)\|_{\mathrm{F}}}{\sqrt{n}}.
\end{aligned}$$

Taking into account that from Theorem 3.5 we have

$$\lim_{n\to\infty} \frac{\|g(A_n) - g(B_n)\|_{\mathrm{F}}}{\sqrt{n}} = 0,$$

we deduce that

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{k=1}^{d_n} (g(\lambda_k(A_n)) - g(\lambda_k(B_n))) \right| = 0,$$

or equivalently,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{d_n} (g(\lambda_k(A_n)) - g(\lambda_k(B_n))) = 0. \qquad \square$$

Theorems 3.5 and 3.6 were introduced for the case in which $\{d_n\} = \{n\}$ in [12] and [8], respectively. Observe that if $\{d_n\} = \{n\}$ then the sequence $\left\{\frac{d_n}{n}\right\} = \{1\}$ is bounded.

4

Block Toeplitz Matrices

4.1 Definition and Examples

We begin this section by reviewing the concept of block Toeplitz matrix.

Definition 4.1. An $m \times n$ *block Toeplitz matrix* with $M \times N$ blocks is an $mM \times nN$ matrix of the form

$$\begin{pmatrix} \mathsf{F}_0 & \mathsf{F}_{-1} & \mathsf{F}_{-2} & \cdots & \mathsf{F}_{1-n} \\ \mathsf{F}_1 & \mathsf{F}_0 & \mathsf{F}_{-1} & \cdots & \mathsf{F}_{2-n} \\ \mathsf{F}_2 & \mathsf{F}_1 & \mathsf{F}_0 & \cdots & \mathsf{F}_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathsf{F}_{m-1} & \mathsf{F}_{m-2} & \mathsf{F}_{m-3} & \cdots & \mathsf{F}_0 \end{pmatrix}, \tag{4.1}$$

where $\mathsf{F}_k \in \mathbb{C}^{M \times N}$ with $1 - n \leq k \leq m - 1$.

When $M = N = 1$ the matrix in (4.1) is simply an $m \times n$ Toeplitz matrix.

A matrix

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \in \mathbb{C}^{mM\times nN}, \tag{4.2}$$

with

$$A_{j,k} \in \mathbb{C}^{M\times N}, \quad 1 \le j \le m, \ 1 \le k \le n,$$

is denoted by $A = (A_{j,k})_{1\le j\le m, 1\le k\le n}$, and by $A = (A_{j,k})_{j,k=1}^{n}$ if $m = n$. The submatrix $A_{j,k}$ is also denoted by $[A]_{j,k}$, and by $a_{j,k}$ when $M = N = 1$. The matrix in (4.2) is an $m \times n$ block Toeplitz matrix with $M \times N$ blocks if $A_{j_1,k_1} = A_{j_2,k_2}$ whenever $j_1 - k_1 = j_2 - k_2$.

We now present two interesting examples of block Toeplitz matrices that are frequently used in Communications, Information Theory and Signal Processing. For those two examples, we need to introduce the following notation. Let $\{\mathbf{x}_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{C}^{N\times 1}$, i.e., $\mathbf{x}_n \in \mathbb{C}^{N\times 1}$ for all $n \in \mathbb{Z}$. If $j, k \in \mathbb{Z}$ with $j \le k$, then

$$\mathbf{x}_{k:j} := \begin{pmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \\ \mathbf{x}_{k-2} \\ \vdots \\ \mathbf{x}_j \end{pmatrix}.$$

Example 4.1. In this first example we consider a discrete-time causal finite impulse response (FIR) multiple-input multiple-output (MIMO) filter, that is, a filter of the form

$$\mathbf{y}_n = \sum_{k=0}^{m} \mathsf{H}_{-k}\mathbf{x}_{n-k}, \quad \forall n \in \mathbb{Z}, \tag{4.3}$$

where the filter taps H_{-k}, with $0 \le k \le m$, are $M \times N$ complex matrices, and the input $\{\mathbf{x}_n\}_{n\in\mathbb{Z}}$ and the output $\{\mathbf{y}_n\}_{n\in\mathbb{Z}}$ of the filter satisfy that $\mathbf{x}_n \in \mathbb{C}^{N\times 1}$ and $\mathbf{y}_n \in \mathbb{C}^{M\times 1}$ for all $n \in \mathbb{Z}$.

From (4.3) we have

$$\mathbf{y}_{n:1} = H_n \mathbf{x}_{n:1-m}, \quad \forall n \in \mathbb{N},$$

where H_n is the $n \times (n+m)$ block Toeplitz matrix with $M \times N$ blocks given by

$$\begin{pmatrix} \mathsf{H}_0 & \mathsf{H}_{-1} & \mathsf{H}_{-2} & \cdots & \mathsf{H}_{-m} & 0_{M\times N} & 0_{M\times N} & \cdots & 0_{M\times N} \\ 0_{M\times N} & \mathsf{H}_0 & \mathsf{H}_{-1} & \cdots & \mathsf{H}_{1-m} & \mathsf{H}_{-m} & 0_{M\times N} & \cdots & 0_{M\times N} \\ 0_{M\times N} & 0_{M\times N} & \mathsf{H}_0 & \cdots & \mathsf{H}_{2-m} & \mathsf{H}_{1-m} & \mathsf{H}_{-m} & \cdots & 0_{M\times N} \\ \vdots & \vdots & \vdots & & & & & \ddots & \vdots \\ 0_{M\times N} & 0_{M\times N} & 0_{M\times N} & \cdots & & & & \cdots & \mathsf{H}_{-m} \end{pmatrix},$$

i.e.,

$$H_n = (\mathsf{H}_{j-k})_{1\le j\le n, 1\le k\le n+m} \tag{4.4}$$

with $\mathsf{H}_{j-k} = 0_{M\times N}$ when $j-k \notin \{-m,\ldots,-1,0\}$.

Thus, matrix representations of discrete-time causal FIR MIMO filters are block Toeplitz.

Example 4.2. In this second example, we consider a vector wide sense stationary (WSS) process. Let $\mathbf{x}_n$ be a (column) random vector of dimension N for all $n \in \mathbb{Z}$. The N-dimensional vector random process[1] $\{\mathbf{x}_n : n \in \mathbb{Z}\}$ is said to be WSS or weakly stationary if it has constant mean, that is,

$$\mathrm{E}(\mathbf{x}_{n_1}) = \mathrm{E}(\mathbf{x}_{n_2}), \quad \forall n_1, n_2 \in \mathbb{Z},$$

and if it satisfies

$$\mathrm{E}(\mathbf{x}_{n_1}\mathbf{x}_{n_2}^*) = \mathrm{E}(\mathbf{x}_{n_3}\mathbf{x}_{n_4}^*) \tag{4.5}$$

whenever $n_1 - n_2 = n_3 - n_4$.

From (4.5) we have

$$\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*) = (\mathsf{X}_{j-k})_{j,k=1}^n, \quad \forall n \in \mathbb{N},$$

where

$$\mathsf{X}_{j-k} = \mathrm{E}(\mathbf{x}_k\mathbf{x}_j^*) \in \mathbb{C}^{N\times N}, \quad \forall j,k \in \mathbb{N}.$$

Thus, the correlation matrix $\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)$ of the random vector $\mathbf{x}_{n:1}$ is an $n \times n$ block Toeplitz matrix with $N \times N$ blocks for all $n \in \mathbb{N}$.

---

[1] Vector random processes are also called multivariate random processes.

4.2 Sequence of Block Toeplitz Matrices Generated by a Continuous Function

Consider a matrix-valued function $F\colon[0,2\pi]\to\mathbb{C}^{M\times N}$. Suppose that F is continuous, that is, the complex function $[F]_{r,s}$, with $[F]_{r,s}(\omega):=[F(\omega)]_{r,s}$, $\omega\in[0,2\pi]$, is continuous for all $1\leq r\leq M$ and $1\leq s\leq N$. Then $\int_0^{2\pi}F(\omega)d\omega$ denotes the $M\times N$ matrix given by

$$\int_0^{2\pi}F(\omega)d\omega:=\left(\int_0^{2\pi}[F(\omega)]_{r,s}d\omega\right)_{1\leq r\leq M,1\leq s\leq N}. \tag{4.6}$$

The following lemma provides three simple properties of the integral introduced in (4.6).

Lemma 4.1. Let $F,G\colon[0,2\pi]\to\mathbb{C}^{M\times N}$ be two continuous matrix-valued functions.

(1) If $\alpha,\beta\in\mathbb{C}$ then

$$\int_0^{2\pi}(\alpha F(\omega)+\beta G(\omega))d\omega=\alpha\int_0^{2\pi}F(\omega)d\omega+\beta\int_0^{2\pi}G(\omega)d\omega.$$

(2) If $A\in\mathbb{C}^{L\times M}$ and $B\in\mathbb{C}^{N\times K}$ then

$$\int_0^{2\pi}AF(\omega)Bd\omega=A\int_0^{2\pi}F(\omega)d\omega B.$$

(3) If $M=N=1$ then

$$\int_0^{2\pi}F(\omega)Ad\omega=\int_0^{2\pi}F(\omega)d\omega A\quad\forall A\in\mathbb{C}^{L\times K}.$$

Proof. (1) Since F and G are continuous, the matrix-valued function $\alpha F(\omega)+\beta G(\omega)$, with $\omega\in[0,2\pi]$, is also continuous. Moreover, we have

$$\begin{aligned}&\int_0^{2\pi}(\alpha F(\omega)+\beta G(\omega))d\omega\\&\quad=\left(\int_0^{2\pi}[\alpha F(\omega)+\beta G(\omega)]_{r,s}d\omega\right)_{1\leq r\leq M,1\leq s\leq N}\\&\quad=\left(\int_0^{2\pi}(\alpha[F(\omega)]_{r,s}+\beta[G(\omega)]_{r,s})d\omega\right)_{1\leq r\leq M,1\leq s\leq N}\end{aligned}$$

$$
\begin{aligned}
&= \left(\alpha \int_0^{2\pi} [F(\omega)]_{r,s} d\omega + \beta \int_0^{2\pi} [G(\omega)]_{r,s} d\omega \right)_{1\leq r\leq M, 1\leq s\leq N} \\
&= \alpha \left(\int_0^{2\pi} [F(\omega)]_{r,s} d\omega \right)_{1\leq r\leq M, 1\leq s\leq N} \\
&\quad + \beta \left(\int_0^{2\pi} [G(\omega)]_{r,s} d\omega \right)_{1\leq r\leq M, 1\leq s\leq N} \\
&= \alpha \int_0^{2\pi} F(\omega) d\omega + \beta \int_0^{2\pi} G(\omega) d\omega.
\end{aligned}
$$

(2) Since F is continuous, the matrix-valued function $AF(\omega)B$, with $\omega \in [0, 2\pi]$, is also continuous. Furthermore,

$$
\begin{aligned}
\int_0^{2\pi} AF(\omega)B d\omega &= \left(\int_0^{2\pi} [AF(\omega)B]_{r,s} d\omega \right)_{1\leq r\leq L, 1\leq s\leq K} \\
&= \left(\int_0^{2\pi} \sum_{k=1}^{M} [A]_{r,k} [F(\omega)B]_{k,s} d\omega \right)_{1\leq r\leq L, 1\leq s\leq K} \\
&= \left(\int_0^{2\pi} \sum_{k=1}^{M} [A]_{r,k} \sum_{l=1}^{N} [F(\omega)]_{k,l} [B]_{l,s} d\omega \right)_{1\leq r\leq L, 1\leq s\leq K} \\
&= \left(\sum_{k=1}^{M} [A]_{r,k} \sum_{l=1}^{N} \int_0^{2\pi} [F(\omega)]_{k,l} d\omega [B]_{l,s} \right)_{1\leq r\leq L, 1\leq s\leq K} \\
&= \left(\sum_{k=1}^{M} [A]_{r,k} \sum_{l=1}^{N} \left[\int_0^{2\pi} F(\omega) d\omega \right]_{k,l} [B]_{l,s} \right)_{1\leq r\leq L, 1\leq s\leq K} \\
&= \left(\sum_{k=1}^{M} [A]_{r,k} \left[\int_0^{2\pi} F(\omega) d\omega B \right]_{k,s} \right)_{1\leq r\leq L, 1\leq s\leq K} \\
&= \left(\left[A \int_0^{2\pi} F(\omega) d\omega B \right]_{r,s} \right)_{1\leq r\leq L, 1\leq s\leq K} \\
&= A \int_0^{2\pi} F(\omega) d\omega B.
\end{aligned}
$$

(3) Since F is continuous, the matrix-valued function $F(\omega)A$, with $\omega \in [0, 2\pi]$, is also continuous. Moreover, we have

$$\begin{aligned}
\int_0^{2\pi} F(\omega)A d\omega &= \left(\int_0^{2\pi} [F(\omega)A]_{r,s} d\omega \right)_{1\le r\le L, 1\le s\le K} \\
&= \left(\int_0^{2\pi} F(\omega)[A]_{r,s} d\omega \right)_{1\le r\le L, 1\le s\le K} \\
&= \left(\int_0^{2\pi} F(\omega) d\omega [A]_{r,s} \right)_{1\le r\le L, 1\le s\le K} \\
&= \int_0^{2\pi} F(\omega) d\omega ([A]_{r,s})_{1\le r\le L, 1\le s\le K} \\
&= \int_0^{2\pi} F(\omega) d\omega A.
\end{aligned}$$

□

We can now define the sequence of block Toeplitz matrices generated by a continuous matrix-valued function.

Definition 4.2. Consider a matrix-valued function of a real variable $F: \mathbb{R} \to \mathbb{C}^{M\times N}$. Suppose that F is continuous and 2π-periodic, that is, the complex function $[F]_{r,s}$, with $[F]_{r,s}(\omega) := [F(\omega)]_{r,s}$, $\omega \in \mathbb{R}$, is continuous and 2π-periodic for all $1 \le r \le M$ and $1 \le s \le N$. For every $n \in \mathbb{N}$, we define $T_n(F)$ as the $n \times n$ block Toeplitz matrix with $M \times N$ blocks given by

$$T_n(F) := (\mathsf{F}_{j-k})_{j,k=1}^{n},$$

where $\{\mathsf{F}_k\}_{k\in\mathbb{Z}}$ is the sequence of Fourier coefficients of F (i.e., $\{[\mathsf{F}_k]_{r,s}\}_{k\in\mathbb{Z}}$ is the sequence of Fourier coefficients of $[F]_{r,s}$ for all $1 \le r \le M$ and $1 \le s \le N$):

$$\mathsf{F}_k = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-k\omega \mathrm{i}} F(\omega) d\omega, \quad \forall k \in \mathbb{Z},$$

where i is the imaginary unit. The sequence $\{T_n(F)\}$ is called the *sequence of block Toeplitz matrices generated by* F, and the continuous matrix-valued function F is called the *generating function* or the *symbol* of the sequence $\{T_n(F)\}$.

When the sequence of block Toeplitz matrices $\{T_n(F)\}$ is the sequence of correlation matrices $\{\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)\}$ of Example 4.2, the generating function F is called the power spectral density of the vector WSS process $\{\mathbf{x}_n : n \in \mathbb{Z}\}$.

The following lemma provides four properties of sequences of block Toeplitz matrices generated by continuous matrix-valued functions.

Lemma 4.2. Let $F,G : \mathbb{R} \to \mathbb{C}^{M\times N}$, and suppose that they are continuous and 2π-periodic. Then

(1) $\{T_n(F^*)\} = \{(T_n(F))^*\}$, where $(F^*)(\omega) := (F(\omega))^*$, $\omega \in \mathbb{R}$.
(2) $\frac{\|T_n(F)\|_F^2}{n} \leq \frac{1}{2\pi}\int_0^{2\pi} \|F(\omega)\|_F^2 d\omega$ for all $n \in \mathbb{N}$.
(3) $\{T_n(\alpha F + \beta G)\} = \{\alpha T_n(F) + \beta T_n(G)\}$ for all $\alpha, \beta \in \mathbb{C}$, where $(\alpha F + \beta G)(\omega) := \alpha F(\omega) + \beta G(\omega)$, $\omega \in \mathbb{R}$.
(4) $\{T_n(F)\} = \{T_n(G)\}$ if and only if $F = G$.

Proof. (1) Since $F : \mathbb{R} \to \mathbb{C}^{M\times N}$ is continuous and 2π-periodic, $F^* : \mathbb{R} \to \mathbb{C}^{N\times M}$ is also continuous and 2π-periodic. Moreover, we have

$$
\begin{aligned}
\{T_n(F^*)\} &= \left\{\left(\left(\frac{1}{2\pi}\int_0^{2\pi}[(F^*)(\omega)]_{r,s}\mathrm{e}^{-(j-k)\omega\mathrm{i}}d\omega\right)_{1\leq r\leq N, 1\leq s\leq M}\right)_{j,k=1}^n\right\} \\
&= \left\{\left(\left(\frac{1}{2\pi}\int_0^{2\pi}[(F(\omega))^*]_{r,s}\mathrm{e}^{-(j-k)\omega\mathrm{i}}d\omega\right)_{1\leq r\leq N, 1\leq s\leq M}\right)_{j,k=1}^n\right\} \\
&= \left\{\left(\left(\frac{1}{2\pi}\int_0^{2\pi}\overline{[F(\omega)]_{s,r}}\mathrm{e}^{-(j-k)\omega\mathrm{i}}d\omega\right)_{1\leq r\leq N, 1\leq s\leq M}\right)_{j,k=1}^n\right\} \\
&= \left\{\left(\left(\overline{\frac{1}{2\pi}\int_0^{2\pi}[F(\omega)]_{s,r}\mathrm{e}^{(j-k)\omega\mathrm{i}}d\omega}\right)_{1\leq r\leq N, 1\leq s\leq M}\right)_{j,k=1}^n\right\} \\
&= \left\{\left(\left(\left(\frac{1}{2\pi}\int_0^{2\pi}[F(\omega)]_{s,r}\mathrm{e}^{(j-k)\omega\mathrm{i}}d\omega\right)_{1\leq s\leq M, 1\leq r\leq N}\right)^*\right)_{j,k=1}^n\right\} \\
&= \left\{\left(([T_n(F)]_{k,j})^*\right)_{j,k=1}^n\right\} \\
&= \left\{\left(([T_n(F)]_{j,k})_{j,k=1}^n\right)^*\right\} = \{(T_n(F))^*\}.
\end{aligned}
$$

(2) Let $\{\mathsf{F}_k\}_{k\in\mathbb{Z}}$ be the sequence of Fourier coefficients of F. From the Parseval theorem (see, e.g., [20, p. 191]) we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} |[F(\omega)]_{r,s}|^2 d\omega = \sum_{k=-\infty}^{\infty} |[\mathsf{F}_k]_{r,s}|^2, \quad 1 \leq r \leq M, \ 1 \leq s \leq N.$$

Consequently,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \|F(\omega)\|_F^2 d\omega &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{r=1}^{M}\sum_{s=1}^{N} |[F(\omega)]_{r,s}|^2 d\omega \\ &= \sum_{r=1}^{M}\sum_{s=1}^{N} \frac{1}{2\pi}\int_0^{2\pi} |[F(\omega)]_{r,s}|^2 d\omega \\ &= \sum_{r=1}^{M}\sum_{s=1}^{N}\sum_{k=-\infty}^{\infty} |[\mathsf{F}_k]_{r,s}|^2 \\ &= \sum_{k=-\infty}^{\infty}\sum_{r=1}^{M}\sum_{s=1}^{N} |[\mathsf{F}_k]_{r,s}|^2 = \sum_{k=-\infty}^{\infty} \|\mathsf{F}_k\|_{\mathrm{F}}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\|T_n(F)\|_{\mathrm{F}}^2}{n} &= \frac{n\|\mathsf{F}_0\|_{\mathrm{F}}^2 + \sum_{k=1}^{n-1}(n-k)\left(\|\mathsf{F}_k\|_{\mathrm{F}}^2 + \|\mathsf{F}_{-k}\|_{\mathrm{F}}^2\right)}{n} \\ &\leq \frac{\sum_{k=-(n-1)}^{n-1} n\|\mathsf{F}_k\|_{\mathrm{F}}^2}{n} = \sum_{k=-(n-1)}^{n-1} \|\mathsf{F}_k\|_{\mathrm{F}}^2 \\ &\leq \sum_{k=-\infty}^{\infty} \|\mathsf{F}_k\|_{\mathrm{F}}^2 = \frac{1}{2\pi}\int_0^{2\pi} \|F(\omega)\|_{\mathrm{F}}^2 d\omega, \quad \forall n \in \mathbb{N}.\end{aligned}$$

(3) Since $F, G : \mathbb{R} \to \mathbb{C}^{M\times N}$ are continuous and 2π-periodic, $\alpha F + \beta G : \mathbb{R} \to \mathbb{C}^{M\times N}$ is also continuous and 2π-periodic. Furthermore, we have

$$\begin{aligned}&\{T_n(\alpha F + \beta G)\} \\ &= \left\{\left(\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-(j-k)\omega \mathrm{i}}((\alpha F + \beta G)(\omega))d\omega\right)_{j,k=1}^{n}\right\} \\ &= \left\{\left(\frac{1}{2\pi}\int_0^{2\pi} \left(\mathrm{e}^{-(j-k)\omega \mathrm{i}}\alpha F(\omega) + \mathrm{e}^{-(j-k)\omega \mathrm{i}}\beta G(\omega)\right) d\omega\right)_{j,k=1}^{n}\right\}\end{aligned}$$

$$= \left\{ \left(\frac{\alpha}{2\pi} \int_0^{2\pi} \mathrm{e}^{-(j-k)\omega \mathrm{i}} F(\omega) d\omega + \frac{\beta}{2\pi} \int_0^{2\pi} \mathrm{e}^{-(j-k)\omega \mathrm{i}} G(\omega) d\omega \right)_{j,k=1}^{n} \right\}$$

$$= \left\{ \alpha \left(\frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-(j-k)\omega \mathrm{i}} F(\omega) d\omega \right)_{j,k=1}^{n} \right.$$

$$\left. + \beta \left(\frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-(j-k)\omega \mathrm{i}} G(\omega) d\omega \right)_{j,k=1}^{n} \right\}$$

$$= \{ \alpha T_n(F) + \beta T_n(G) \}.$$

(4) If $\{T_n(F)\} = \{T_n(G)\}$ then F and G have the same sequence of Fourier coefficients, i.e., the continuous 2π-periodic functions $[F]_{r,s}$ and $[G]_{r,s}$ have the same sequence of Fourier coefficients for all $1 \leq r \leq M$ and $1 \leq s \leq N$. Since from the Fejér theorem (see, e.g., [20, p. 199]) two continuous 2π-periodic functions with the same sequence of Fourier coefficients are equal, $[F]_{r,s} = [G]_{r,s}$ for all $1 \leq r \leq M$ and $1 \leq s \leq N$. Consequently, $F = G$. □

We now present a result that gives an upper bound for the singular values of the block Toeplitz matrices generated by a continuous matrix-valued function.

Theorem 4.3. Let $F : \mathbb{R} \to \mathbb{C}^{M \times N}$ be a matrix-valued function which is continuous and 2π-periodic. Then

$$\sigma_1(T_n(F)) \leq \sigma_1(F), \quad \forall n \in \mathbb{N}, \tag{4.7}$$

where

$$\sigma_1(F) := \sup_{\omega \in [0,2\pi]} \sigma_1(F(\omega)) < \infty.$$

Proof. We begin by proving that $\sigma_1(F) < \infty$. It is well known that if A is an $m \times n$ matrix, the m eigenvalues of the Hermitian matrix AA^* are given by $(\sigma_1(A))^2, \ldots, (\sigma_m(A))^2$ when $m \leq n$, and by $(\sigma_1(A))^2, \ldots, (\sigma_n(A))^2, 0, \ldots, 0$ when $n < m$. Consequently,

$$\operatorname{tr}(F(\omega)(F(\omega))^*) = \sum_{k=1}^{M} \lambda_k(F(\omega)(F(\omega))^*) = \sum_{k=1}^{\min\{M,N\}} (\sigma_k(F(\omega)))^2$$

for all $\omega \in \mathbb{R}$. Therefore,

$$\begin{aligned}(\sigma_1(F(\omega_0)))^2 &\leq \operatorname{tr}(F(\omega_0)(F(\omega_0))^*) \\ &\leq \max_{\omega\in[0,2\pi]} \operatorname{tr}(F(\omega)(F(\omega))^*), \quad \forall \omega_0 \in [0,2\pi],\end{aligned}$$

where the existence of $\max_{\omega\in[0,2\pi]} \operatorname{tr}(F(\omega)(F(\omega))^*)$ is guaranteed by the continuity of the real function $\operatorname{tr}(F(\omega)(F(\omega))^*)$ on the closed interval $[0,2\pi]$. Hence,

$$\sigma_1(F(\omega_0)) \leq \sqrt{\max_{\omega\in[0,2\pi]} \operatorname{tr}(F(\omega)(F(\omega))^*)}, \quad \forall \omega_0 \in [0,2\pi],$$

and this guarantees that $\sigma_1(F) < \infty$.

We now proceed to prove (4.7). Fix $n \in \mathbb{N}$, and let $T_n(F) = U\Sigma V^*$ be a singular value decomposition of $T_n(F)$. Since U and V are unitary, we obtain

$$\begin{aligned}\sigma_1(T_n(F)) &= [\Sigma]_{1,1} = [U^* T_n(F) V]_{1,1} = \sum_{h=1}^{n} [U^*]_{1,h} [T_n(F)V]_{h,1} \\ &= \sum_{h=1}^{n} [U^*]_{1,h} \sum_{l=1}^{n} [T_n(F)]_{h,l} [V]_{l,1} \\ &= \sum_{h=1}^{n} [U]_{h,1}^* \sum_{l=1}^{n} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-(h-l)\omega \mathrm{i}} F(\omega) d\omega [V]_{l,1} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{h=1}^{n} [U]_{h,1}^* \sum_{l=1}^{n} \mathrm{e}^{(-h+l)\omega \mathrm{i}} F(\omega) [V]_{l,1} d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{h=1}^{n} \mathrm{e}^{h\omega \mathrm{i}} [U]_{h,1} \right)^* F(\omega) \left(\sum_{l=1}^{n} \mathrm{e}^{l\omega \mathrm{i}} [V]_{l,1} \right) d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left((S(\omega))^* \sum_{h=1}^{n} \mathrm{e}^{h\omega \mathrm{i}} [U]_{h,1} \right)^* \Lambda(\omega) \\ &\quad \left((W(\omega))^* \sum_{l=1}^{n} \mathrm{e}^{l\omega \mathrm{i}} [V]_{l,1} \right) d\omega\end{aligned}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{\min\{M,N\}}\sigma_j(F(\omega))\left[\left((S(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[U]_{h,1}\right)^*\right]_{1,j}$$
$$\left[(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}\right]_{j,1} d\omega$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{j=1}^{\min\{M,N\}}\sigma_j(F(\omega))\left[\left((S(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[U]_{h,1}\right)^*\right]_{1,j}\right.$$
$$\left.\left[(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}\right]_{j,1}\right| d\omega, \tag{4.8}$$

where $F(\omega) = S(\omega)\Lambda(\omega)(W(\omega))^*$ is a singular value decomposition of the matrix $F(\omega)$ for all $\omega \in [0, 2\pi]$. Applying the Cauchy–Schwarz inequality (see, e.g., [1, p. 14]) we have

$$\left|\sum_{j=1}^{\min\{M,N\}}\sigma_j(F(\omega))\left[\left((S(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[U]_{h,1}\right)^*\right]_{1,j}\left[(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}\right]_{j,1}\right|$$
$$\leq \sum_{j=1}^{\min\{M,N\}}\sigma_j(F(\omega))\left|\left[\left((S(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[U]_{h,1}\right)^*\right]_{1,j}\right|$$
$$\left|\left[(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}\right]_{j,1}\right|$$
$$\leq \sigma_1(F)\sum_{j=1}^{\min\{M,N\}}\left|\left[\left((S(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[U]_{h,1}\right)^*\right]_{1,j}\right|$$
$$\left|\left[(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}\right]_{j,1}\right|$$
$$\leq \sigma_1(F)\sqrt{\sum_{j=1}^{\min\{M,N\}}\left|\left[\left((S(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[U]_{h,1}\right)^*\right]_{1,j}\right|^2}$$
$$\sqrt{\sum_{j=1}^{\min\{M,N\}}\left|\left[(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}\right]_{j,1}\right|^2}$$

$$\leq \sigma_1(F)\sqrt{\sum_{j=1}^{M}\left|\left[\left((S(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[U]_{h,1}\right)^*\right]_{1,j}\right|^2}$$

$$\sqrt{\sum_{j=1}^{N}\left|\left[(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}\right]_{j,1}\right|^2}, \quad \forall\omega\in[0,2\pi]. \tag{4.9}$$

From (4.8) and (4.9), and using the Cauchy–Schwarz inequality (see, e.g., [20, p. 139]) we obtain

$$\sigma_1(T_n(F)) \leq \frac{\sigma_1(F)}{2\pi}\sqrt{\int_0^{2\pi}\sum_{j=1}^{M}\left|\left[\left((S(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[U]_{h,1}\right)^*\right]_{1,j}\right|^2 d\omega}$$

$$\sqrt{\int_0^{2\pi}\sum_{j=1}^{N}\left|\left[(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}\right]_{j,1}\right|^2 d\omega}$$

$$= \sigma_1(F)\sqrt{\frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{M}\left|\left[(S(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[U]_{h,1}\right]_{j,1}\right|^2 d\omega}$$

$$\sqrt{\frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{N}\left|\left[(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}\right]_{j,1}\right|^2 d\omega}. \tag{4.10}$$

On the other hand, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{N}\left|\left[(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}\right]_{j,1}\right|^2 d\omega$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{N}\left[\left((W(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[V]_{h,1}\right)^*\right]_{1,j}$$

$$\left[(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}\right]_{j,1} d\omega$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\left((W(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[V]_{h,1}\right)^*(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}d\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{h=1}^{n}\mathrm{e}^{-h\omega\mathrm{i}}[V]_{h,1}^*W(\omega)(W(\omega))^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}d\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{h=1}^{n}\mathrm{e}^{-h\omega\mathrm{i}}[V]_{h,1}^*\sum_{l=1}^{n}\mathrm{e}^{l\omega\mathrm{i}}[V]_{l,1}d\omega\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{h=1}^{n}[V]_{h,1}^*\sum_{l=1}^{n}\mathrm{e}^{(-h+l)\omega\mathrm{i}}[V]_{l,1}d\omega\\
&= \sum_{h=1}^{n}[V]_{h,1}^*\sum_{l=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{(-h+l)\omega\mathrm{i}}d\omega[V]_{l,1} = \sum_{h=1}^{n}[V]_{h,1}^*[V]_{h,1}\\
&= \sum_{h=1}^{n}[V^*]_{1,h}[V]_{h,1} = [V^*V]_{1,1} = [I_{nN}]_{1,1} = 1, \qquad (4.11)
\end{aligned}
$$

because

$$
\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{m\omega\mathrm{i}}d\omega = \begin{cases}1, & \text{if } m = 0,\\ 0, & \text{if } m \text{ is a non-zero integer number.}\end{cases}
$$

By reasoning as in (4.11) we also obtain

$$
\frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{M}\left|\left[(S(\omega))^*\sum_{h=1}^{n}\mathrm{e}^{h\omega\mathrm{i}}[U]_{h,1}\right]_{j,1}\right|^2 d\omega = 1. \qquad (4.12)
$$

Using (4.10), (4.11) and (4.12) yields (4.7). □

Observe that if $M = N = 1$ then

$$
\sigma_1(F) = \max_{\omega\in[0,2\pi]}|F(\omega)|.
$$

The next result provides an upper bound and a lower bound for the eigenvalues of the block Toeplitz matrices generated by a continuous matrix-valued function which is Hermitian.

Theorem 4.4. Consider a matrix-valued function $F: \mathbb{R} \to \mathbb{C}^{N \times N}$ which is continuous and 2π-periodic.

(1) F is Hermitian (that is, $F^* = F$) if and only if $T_n(F)$ is Hermitian for all $n \in \mathbb{N}$.
(2) If F is Hermitian then

$$\inf F \leq \lambda_k(T_n(F)) \leq \sup F, \quad 1 \leq k \leq nN, \ n \in \mathbb{N}, \quad (4.13)$$

where

$$\inf F := \inf_{\substack{1 \leq k \leq N \\ \omega \in [0,2\pi]}} \lambda_k(F(\omega)) > -\infty,$$

and

$$\sup F := \sup_{\substack{1 \leq k \leq N \\ \omega \in [0,2\pi]}} \lambda_k(F(\omega)) < \infty.$$

Proof. (1) From property (1) of Lemma 4.2, if F is Hermitian we have

$$(T_n(F))^* = T_n(F^*) = T_n(F), \quad \forall n \in \mathbb{N}.$$

The reciprocal is also true because if $\{T_n(F)\} = \{(T_n(F))^*\} = \{T_n(F^*)\}$ applying property (4) of Lemma 4.2 yields $F = F^*$.
(2) We begin by proving that $\inf F > -\infty$ and $\sup F < \infty$. Since

$$\operatorname{tr}((F(\omega))^2) = \sum_{k=1}^{N} (\lambda_k(F(\omega)))^2 = \sum_{k=1}^{N} |\lambda_k(F(\omega))|^2, \quad \forall \omega \in \mathbb{R},$$

we obtain

$$|\lambda_k(F(\omega_0))|^2 \leq \max_{\omega \in [0,2\pi]} \operatorname{tr}((F(\omega))^2), \quad \omega_0 \in [0,2\pi], \quad 1 \leq k \leq N,$$

where the existence of $\max_{\omega \in [0,2\pi]} \operatorname{tr}((F(\omega))^2)$ is guaranteed by the continuity of the real function $\operatorname{tr}((F(\omega))^2)$ on the closed interval $[0,2\pi]$. Hence, if $\omega_0 \in [0,2\pi]$ and $1 \leq k \leq N$ then

$$-\sqrt{\max_{\omega \in [0,2\pi]} \operatorname{tr}((F(\omega))^2)} \leq \lambda_k(F(\omega_0)) \leq \sqrt{\max_{\omega \in [0,2\pi]} \operatorname{tr}((F(\omega))^2)},$$

and this guarantees that $\inf F > -\infty$ and $\sup F < \infty$.

We now proceed to prove (4.13). Fix $n \in \mathbb{N}$, and let $T_n(F) = U\text{diag}(\lambda_1(T_n(F)), \ldots, \lambda_{nN}(T_n(F)))U^{-1}$ be an eigenvalue decomposition of the Hermitian matrix $T_n(F)$, where the eigenvector matrix U is unitary. If $1 \leq k \leq nN$ then

$$\begin{aligned}
\lambda_k(T_n(F)) &= [U^* T_n(F) U]_{k,k} = \sum_{h=1}^{n} [U^*]_{k,h} [T_n(F) U]_{h,k} \\
&= \sum_{h=1}^{n} [U^*]_{k,h} \sum_{l=1}^{n} [T_n(F)]_{h,l} [U]_{l,k} \\
&= \sum_{h=1}^{n} [U]_{h,k}^* \sum_{l=1}^{n} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-(h-l)\omega \mathrm{i}} F(\omega) d\omega [U]_{l,k} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{h=1}^{n} [U]_{h,k}^* \sum_{l=1}^{n} \mathrm{e}^{(-h+l)\omega \mathrm{i}} F(\omega) [U]_{l,k} d\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{h=1}^{n} \mathrm{e}^{h\omega \mathrm{i}} [U]_{h,k} \right)^* F(\omega) \left(\sum_{l=1}^{n} \mathrm{e}^{l\omega \mathrm{i}} [U]_{l,k} \right) d\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left((W(\omega))^* \sum_{h=1}^{n} \mathrm{e}^{h\omega \mathrm{i}} [U]_{h,k} \right)^* \\
&\quad \text{diag}(\lambda_1(F(\omega)), \ldots, \lambda_N(F(\omega))) \left((W(\omega))^* \sum_{l=1}^{n} \mathrm{e}^{l\omega \mathrm{i}} [U]_{l,k} \right) d\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^{N} \lambda_j(F(\omega)) \left[\left((W(\omega))^* \sum_{h=1}^{n} \mathrm{e}^{h\omega \mathrm{i}} [U]_{h,k} \right)^* \right]_{1,j} \\
&\quad \left[(W(\omega))^* \sum_{l=1}^{n} \mathrm{e}^{l\omega \mathrm{i}} [U]_{l,k} \right]_{j,1} d\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^{N} \lambda_j(F(\omega)) \left| \left[(W(\omega))^* \sum_{l=1}^{n} \mathrm{e}^{l\omega \mathrm{i}} [U]_{l,k} \right]_{j,1} \right|^2 d\omega,
\end{aligned}$$

where $F(\omega) = W(\omega)\text{diag}(\lambda_1(F(\omega)), \ldots, \lambda_N(F(\omega)))(W(\omega))^{-1}$ is an eigenvalue decomposition of the Hermitian matrix $F(\omega)$, and the eigenvector matrix $W(\omega)$ is unitary for all $\omega \in [0, 2\pi]$. Consequently,

$$\inf F \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^{N} \left| \left[(W(\omega))^* \sum_{l=1}^{n} \mathrm{e}^{l\omega \mathrm{i}} [U]_{l,k} \right]_{j,1} \right|^2 d\omega \leq \lambda_k(T_n(F))$$

$$\leq \sup F \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^{N} \left| \left[(W(\omega))^* \sum_{l=1}^{n} \mathrm{e}^{l\omega \mathrm{i}} [U]_{l,k} \right]_{j,1} \right|^2 d\omega \qquad (4.14)$$

for all $1 \leq k \leq nN$. By reasoning as in (4.11) we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^{N} \left| \left[(W(\omega))^* \sum_{l=1}^{n} \mathrm{e}^{l\omega \mathrm{i}} [U]_{l,k} \right]_{j,1} \right|^2 d\omega = 1 \qquad (4.15)$$

for all $1 \leq k \leq nN$. Using (4.14) and (4.15) yields (4.13). □

Observe that if $N = 1$ then

$$\inf F = \min_{\omega \in [0, 2\pi]} F(\omega),$$

and

$$\sup F = \max_{\omega \in [0, 2\pi]} F(\omega).$$

The product of block Toeplitz matrices is not, in general, a block Toeplitz matrix:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

To end this subsection, we give a non-trivial example of block Toeplitz matrix that can be obtained as the product of block Toeplitz matrices.

Lemma 4.5. Consider a matrix-valued function $F \colon \mathbb{R} \to \mathbb{C}^{N \times N}$ which is continuous and 2π-periodic. For every $n \in \mathbb{N}$, let H_n be the $n \times (n+m)$ block Toeplitz matrix with $M \times N$ blocks in (4.4). Then

$$\{T_n(HFH^*)\} = \{H_n T_{n+m}(F) H_n^*\},$$

where $(HFH^*)(\omega) = H(\omega)F(\omega)(H(\omega))^*$ and $H(\omega) = \sum_{k=-m}^{0} \mathrm{e}^{k\omega \mathrm{i}} \mathsf{H}_k$ for all $\omega \in \mathbb{R}$.

Proof. Since $F\colon \mathbb{R} \to \mathbb{C}^{N\times N}$ and $H : \mathbb{R} \to \mathbb{C}^{M\times N}$ are continuous and 2π-periodic, $HFH^* : \mathbb{R} \to \mathbb{C}^{M\times M}$ is also continuous and 2π-periodic. For every $n \in \mathbb{N}$ we have

$$\begin{aligned}
[T_n(HFH^*)]_{j,k} &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-(j-k)\omega \mathrm{i}} H(\omega)F(\omega)(H(\omega))^* d\omega \\
&= \sum_{h=0}^{m} \mathsf{H}_{-h}\left(\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{(-(j-k)-h)\omega \mathrm{i}} F(\omega)(H(\omega))^* d\omega\right) \\
&= \sum_{h=0}^{m} \mathsf{H}_{-h}\sum_{l=0}^{m}\left(\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{(-(j-k)-h+l)\omega \mathrm{i}} F(\omega) d\omega\right)\mathsf{H}_{-l}^* \\
&= \sum_{h=0}^{m} \mathsf{H}_{-h}\sum_{l=0}^{m} \mathsf{F}_{j-k+h-l}\mathsf{H}_{-l}^* \\
&= \sum_{h=0}^{m} [H_n]_{j,j+h}\sum_{l=0}^{m} [T_{n+m}(F)]_{j+h,k+l}[H_n]_{k,k+l}^* \\
&= \sum_{\hat{h}=j}^{j+m} [H_n]_{j,\hat{h}}\sum_{\hat{l}=k}^{k+m} [T_{n+m}(F)]_{\hat{h},\hat{l}}[H_n]_{k,\hat{l}}^* \\
&= \sum_{\hat{h}=j}^{j+m} [H_n]_{j,\hat{h}}\sum_{\hat{l}=1}^{n+m} [T_{n+m}(F)]_{\hat{h},\hat{l}}[H_n]_{k,\hat{l}}^* \\
&= \sum_{\hat{h}=j}^{j+m} [H_n]_{j,\hat{h}}\sum_{\hat{l}=1}^{n+m} [T_{n+m}(F)]_{\hat{h},\hat{l}}[H_n^*]_{\hat{l},k} \\
&= \sum_{\hat{h}=j}^{j+m} [H_n]_{j,\hat{h}}[T_{n+m}(F)H_n^*]_{\hat{h},k} \\
&= \sum_{\hat{h}=1}^{n+m} [H_n]_{j,\hat{h}}[T_{n+m}(F)H_n^*]_{\hat{h},k} \\
&= [H_n T_{n+m}(F)H_n^*]_{j,k}, \qquad 1 \le j,k \le n,
\end{aligned}$$

where $\{\mathsf{F}_k\}_{k\in\mathbb{Z}}$ is the sequence of Fourier coefficients of F. □

4.3 Examples of Continuous Generating Functions

We now present three interesting examples of continuous symbols.

Example 4.3. A simple example of continuous symbols are the trigonometric polynomials. An $M \times N$ *trigonometric polynomial* is a matrix-valued function of a real variable $F\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ of the form

$$F(\omega) = \sum_{h=-m}^{m} \mathrm{e}^{h\omega\mathrm{i}} \mathsf{A}_h, \quad \forall \omega \in \mathbb{R}, \tag{4.16}$$

where $m \in \mathbb{N} \cup \{0\}$ and $\mathsf{A}_h \in \mathbb{C}^{M\times N}$ with $|h| \leq m$. Obviously, the function F in (4.16) is continuous and 2π-periodic. Furthermore, its sequence of Fourier coefficients $\{\mathsf{F}_k\}_{k\in\mathbb{Z}}$ is given by

$$\begin{aligned}
\mathsf{F}_k &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-k\omega\mathrm{i}} \sum_{h=-m}^{m} \mathrm{e}^{h\omega\mathrm{i}} \mathsf{A}_h d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{h=-m}^{m} \mathrm{e}^{(h-k)\omega\mathrm{i}} \mathsf{A}_h d\omega \\
&= \sum_{h=-m}^{m} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{(h-k)\omega\mathrm{i}} d\omega \mathsf{A}_h = \begin{cases} \mathsf{A}_k, & \text{if } |k| \leq m, \\ 0_{M\times N}, & \text{if } |k| > m. \end{cases}
\end{aligned}$$

Consequently, the block Toeplitz matrix $T_n(F) = (\mathsf{F}_{j-k})_{j,k=1}^{n}$ generated by the trigonometric polynomial F is banded[2] with bandwidth $\max\{(m+1)M-1, (m+1)N-1\}$ for all $n > m+1$.

Example 4.4. Let $\{\mathsf{F}_k\}_{k\in\mathbb{Z}}$ be the sequence of Fourier coefficients of a matrix-valued function $F\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ which is continuous and 2π-periodic. We say that F is in the Wiener class if the complex function $[F]_{r,s}$ is in the Wiener class for all $1 \leq r \leq M$ and $1 \leq s \leq N$, i.e., if the

[2] $A = (a_{j,k})_{1\leq j\leq m, 1\leq k\leq n} \in \mathbb{C}^{m\times n}$ is banded with bandwidth $b \in \mathbb{N} \cup \{0\}$ if $a_{j,k} = 0$ whenever $|j-k| > b$. Banded matrices are also called band matrices.

sequence of Fourier coefficients $\{[\mathsf{F}_k]_{r,s}\}_{k\in\mathbb{Z}}$ of $[F]_{r,s}$ satisfies

$$\sum_{k=-\infty}^{\infty} |[\mathsf{F}_k]_{r,s}| < \infty$$

for all $1 \leq r \leq M$ and $1 \leq s \leq N$.

In Appendix B, we present the Wiener class in more detail.

Example 4.5. We prove here that continuous functions of a continuous symbol which is Hermitian are also examples of continuous symbols. Consider a Hermitian matrix-valued function $F\colon \mathbb{R} \to \mathbb{C}^{N\times N}$ which is continuous and 2π-periodic. If $g \in C[\inf F, \sup F]$, since F is 2π-periodic, the matrix-valued function $g(F)\colon \mathbb{R} \to \mathbb{C}^{N\times N}$ with $(g(F))(\omega) := g(F(\omega))$, $\omega \in \mathbb{R}$, is also 2π-periodic. To finish this example we need to prove that $g(F)$ is continuous. Let

$$F(\omega) = W(\omega)\text{diag}(\lambda_1(F(\omega)), \ldots, \lambda_N(F(\omega)))(W(\omega))^{-1}$$

be an eigenvalue decomposition of the Hermitian matrix $F(\omega)$, where the eigenvector matrix $W(\omega)$ is unitary for all $\omega \in \mathbb{R}$. As g is a continuous complex function on $[\inf F, \sup F] \subset \mathbb{R}$, from the Stone–Weierstrass theorem there exists a sequence of polynomials $\{p_n(x)\}$ with coefficients in $\mathbb{C}$, that converges uniformly to g on $[\inf F, \sup F]$. Thus, for every $\epsilon \in (0, \infty)$ there exists $n_0 \in \mathbb{N}$ such that

$$|p_n(x) - g(x)| < \epsilon, \quad x \in [\inf F, \sup F], \;\; n \geq n_0.$$

Applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&|[p_n(F(\omega))]_{j,k} - [g(F(\omega))]_{j,k}| \\
&\quad = |[p_n(F(\omega)) - g(F(\omega))]_{j,k}| = |[(p_n - g)(F(\omega))]_{j,k}| \\
&\quad = |[W(\omega)\text{diag}((p_n - g)(\lambda_1(F(\omega))), \ldots, (p_n - g)(\lambda_N(F(\omega)))) \\
&\qquad (W(\omega))^*]_{j,k}| \\
&\quad = \left| \sum_{l=1}^{N} (p_n - g)(\lambda_l(F(\omega)))[W(\omega)]_{j,l}[(W(\omega))^*]_{l,k} \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{l=1}^{N} (p_n - g)(\lambda_l(F(\omega)))[W(\omega)]_{j,l}\overline{[W(\omega)]_{k,l}} \right| \\
&\leq \sum_{l=1}^{N} |(p_n - g)(\lambda_l(F(\omega)))||[W(\omega)]_{j,l}| \left| \overline{[W(\omega)]_{k,l}} \right| \\
&\leq \epsilon \sum_{l=1}^{N} |[W(\omega)]_{j,l}||[W(\omega)]_{k,l}| \leq \epsilon \sqrt{\sum_{l=1}^{N} |[W(\omega)]_{j,l}|^2} \sqrt{\sum_{l=1}^{N} |[W(\omega)]_{k,l}|^2} \\
&= \epsilon \sqrt{\sum_{l=1}^{N} [W(\omega)]_{j,l}\overline{[W(\omega)]_{j,l}}} \sqrt{\sum_{l=1}^{N} [W(\omega)]_{k,l}\overline{[W(\omega)]_{k,l}}} \\
&= \epsilon \sqrt{\sum_{l=1}^{N} [W(\omega)]_{j,l}[(W(\omega))^*]_{l,j}} \sqrt{\sum_{l=1}^{N} [W(\omega)]_{k,l}[(W(\omega))^*]_{l,k}} \\
&= \epsilon \sqrt{[W(\omega)(W(\omega))^*]_{j,j}} \sqrt{[W(\omega)(W(\omega))^*]_{k,k}} = \epsilon \sqrt{[I_N]_{j,j}} \sqrt{[I_N]_{k,k}} \\
&= \epsilon, \qquad \omega \in \mathbb{R}, \quad 1 \leq j,k \leq N, \quad n \geq n_0. \qquad (4.17)
\end{aligned}
$$

Fix $1 \leq j,k \leq N$. On the one hand, the complex function $[p_n(F(\omega))]_{j,k}$ is continuous on $\mathbb{R}$ for all $n \in \mathbb{N}$, because F is continuous and the $p_n(x)$ are polynomials. On the other hand, from (4.17) the sequence $\{[p_n(F(\omega))]_{j,k}\}$ converges uniformly to the complex function $[g(F(\omega))]_{j,k}$ on $\mathbb{R}$. Consequently, $[g(F(\omega))]_{j,k}$ is also continuous on $\mathbb{R}$. This finishes the proof of the continuity of $g(F)$.

5

Block Circulant Matrices

5.1 Definition

This section is devoted to a special type of block Toeplitz matrices: block circulant matrices. This type of matrices will be used in Section 6 to study the asymptotic behaviour of the block Toeplitz matrices generated by a continuous matrix-valued function.

We begin by reviewing the concept of a block circulant matrix.

Definition 5.1. An $n \times n$ *block circulant matrix* with $M \times N$ blocks is an $nM \times nN$ matrix of the form

$$\begin{pmatrix} \mathsf{C}_0 & \mathsf{C}_{-1} & \mathsf{C}_{-2} & \cdots & \mathsf{C}_{1-n} \\ \mathsf{C}_{1-n} & \mathsf{C}_0 & \mathsf{C}_{-1} & \cdots & \mathsf{C}_{2-n} \\ \mathsf{C}_{2-n} & \mathsf{C}_{1-n} & \mathsf{C}_0 & \cdots & \mathsf{C}_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathsf{C}_{-1} & \mathsf{C}_{-2} & \mathsf{C}_{-3} & \cdots & \mathsf{C}_0 \end{pmatrix}, \tag{5.1}$$

where $\mathsf{C}_k \in \mathbb{C}^{M\times N}$ with $1-n \leq k \leq 0$. Thus, an $nM \times nN$ matrix $C = (C_{j,k})_{j,k=1}^{n}$ is an $n \times n$ block circulant matrix with $M \times N$ blocks if $C_{j_1,k_1} = C_{j_2,k_2}$, whenever $(j_1-k_1)-(j_2-k_2) \in \{-n, 0, n\}$.

The matrix in (5.1) is denoted by $\mathrm{circ}(\mathsf{C}_0, \mathsf{C}_{-1}, \ldots, \mathsf{C}_{1-n})$. When $M = N = 1$ this matrix is simply an $n \times n$ circulant matrix.

The next result, which can be found in [13], characterizes block circulant matrices.

Lemma 5.1. Let $C = (C_{j,k})_{j,k=1}^{n} \in \mathbb{C}^{nM \times nN}$, then the following statements are equivalent:

(1) C is an $n \times n$ block circulant matrix with $M \times N$ blocks.
(2) There exist $A_1, \ldots, A_n \in \mathbb{C}^{M \times N}$ such that

$$C = (V_n \otimes I_M)\mathrm{diag}(A_1, \ldots, A_n)(V_n \otimes I_N)^*,$$

where $\otimes$ is the Kronecker product, $\mathrm{diag}(A_1, \ldots, A_n) = \mathrm{diag}_{1 \le k \le n}(A_k) := (\delta_{j,k} A_j)_{j,k=1}^{n}$, $\delta_{j,k}$ is the Kronecker delta, and V_n is the $n \times n$ Fourier unitary matrix:

$$[V_n]_{j,k} := \frac{1}{\sqrt{n}} \mathrm{e}^{-\frac{2\pi(j-1)(k-1)}{n}\mathrm{i}}, \quad 1 \le j,k \le n.$$

Furthermore, if C is a block circulant matrix then

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} = \sqrt{n}(V_n \otimes I_M)^* \begin{pmatrix} C_{1,1} \\ C_{2,1} \\ \vdots \\ C_{n,1} \end{pmatrix}. \tag{5.2}$$

Proof. (1)⇒(2) $V_n \otimes I_M$ is unitary because

$$\begin{aligned} (V_n \otimes I_M)(V_n \otimes I_M)^* &= (V_n \otimes I_M)(V_n^* \otimes I_M^*) \\ &= V_n V_n^* \otimes I_M I_M^* = I_n \otimes I_M = I_{nM}, \end{aligned}$$

and

$$\begin{aligned} (V_n \otimes I_M)^*(V_n \otimes I_M) &= (V_n^* \otimes I_M^*)(V_n \otimes I_M) \\ &= V_n^* V_n \otimes I_M^* I_M = I_n \otimes I_M = I_{nM}. \end{aligned}$$

Analogously, it can be proved that $V_n \otimes I_N$ is unitary. Consequently, if $D = (V_n \otimes I_M)^* C (V_n \otimes I_N) \in \mathbb{C}^{nM \times nN}$, then $C = (V_n \otimes I_M) D (V_n \otimes I_N)^*$. We now proceed to prove that $D = \mathrm{diag}(A_1, \ldots, A_n)$ with

$A_1, \ldots, A_n$ satisfying (5.2). For every $1 \leq j, k \leq n$, we obtain

$$\begin{aligned}
D_{j,k} &= [(V_n \otimes I_M)^* C (V_n \otimes I_N)]_{j,k} = [(V_n^* \otimes I_M^*) C (V_n \otimes I_N)]_{j,k} \\
&= [(V_n^* \otimes I_M) C (V_n \otimes I_N)]_{j,k} = \sum_{h=1}^{n} [V_n^* \otimes I_M]_{j,h} [C(V_n \otimes I_N)]_{h,k} \\
&= \sum_{h=1}^{n} [V_n^* \otimes I_M]_{j,h} \sum_{l=1}^{n} C_{h,l} [V_n \otimes I_N]_{l,k} \\
&= \sum_{h=1}^{n} [V_n^*]_{j,h} I_M \sum_{l=1}^{n} C_{h,l} ([V_n]_{l,k} I_N) = \sum_{h=1}^{n} \overline{[V_n]_{h,j}} \sum_{l=1}^{n} [V_n]_{l,k} C_{h,l} \\
&= \sum_{h=1}^{n} \sum_{l=1}^{n} \overline{[V_n]_{h,j}} [V_n]_{l,k} C_{h,l} \\
&= \sum_{h=1}^{n} \sum_{l=1}^{h} \overline{[V_n]_{h,j}} [V_n]_{l,k} C_{h,l} + \sum_{h=1}^{n} \sum_{l=h+1}^{n} \overline{[V_n]_{h,j}} [V_n]_{l,k} C_{h,l} \\
&= \sum_{h=1}^{n} \sum_{l=1}^{h} \overline{[V_n]_{h,j}} [V_n]_{l,k} C_{h-l+1,1} + \sum_{h=1}^{n} \sum_{l=h+1}^{n} \overline{[V_n]_{h,j}} [V_n]_{l,k} C_{n+h-l+1,1} \\
&= \sum_{h=1}^{n} \sum_{\hat{l}=1}^{h} \overline{[V_n]_{h,j}} [V_n]_{h-\hat{l}+1,k} C_{\hat{l},1} + \sum_{h=1}^{n} \sum_{\hat{l}=h+1}^{n} \overline{[V_n]_{h,j}} [V_n]_{n+h-\hat{l}+1,k} C_{\hat{l},1} \\
&= \sum_{h=1}^{n} \sum_{\hat{l}=1}^{h} \frac{1}{n} \mathrm{e}^{\frac{2\pi(h-1)(j-1)}{n}\mathrm{i}} \mathrm{e}^{-\frac{2\pi(h-\hat{l})(k-1)}{n}\mathrm{i}} C_{\hat{l},1} \\
&\quad + \sum_{h=1}^{n} \sum_{\hat{l}=h+1}^{n} \frac{1}{n} \mathrm{e}^{\frac{2\pi(h-1)(j-1)}{n}\mathrm{i}} \mathrm{e}^{-\frac{2\pi(n+h-\hat{l})(k-1)}{n}\mathrm{i}} C_{\hat{l},1} \\
&= \sum_{h=1}^{n} \sum_{\hat{l}=1}^{h} \frac{1}{n} \mathrm{e}^{\frac{2\pi(h-1)(j-1)}{n}\mathrm{i}} \mathrm{e}^{-\frac{2\pi(h-\hat{l})(k-1)}{n}\mathrm{i}} C_{\hat{l},1} \\
&\quad + \sum_{h=1}^{n} \sum_{\hat{l}=h+1}^{n} \frac{1}{n} \mathrm{e}^{\frac{2\pi(h-1)(j-1)}{n}\mathrm{i}} \mathrm{e}^{-\frac{2\pi(h-\hat{l})(k-1)}{n}\mathrm{i}} \mathrm{e}^{-2\pi(k-1)\mathrm{i}} C_{\hat{l},1} \\
&= \sum_{h=1}^{n} \sum_{\hat{l}=1}^{h} \frac{1}{n} \mathrm{e}^{\frac{2\pi(h-1)(j-1)}{n}\mathrm{i}} \mathrm{e}^{-\frac{2\pi(h-\hat{l})(k-1)}{n}\mathrm{i}} C_{\hat{l},1} \\
&\quad + \sum_{h=1}^{n} \sum_{\hat{l}=h+1}^{n} \frac{1}{n} \mathrm{e}^{\frac{2\pi(h-1)(j-1)}{n}\mathrm{i}} \mathrm{e}^{-\frac{2\pi(h-\hat{l})(k-1)}{n}\mathrm{i}} C_{\hat{l},1} \\
&= \sum_{h=1}^{n} \sum_{\hat{l}=1}^{n} \frac{1}{n} \mathrm{e}^{\frac{2\pi(h-1)(j-1)}{n}\mathrm{i}} \mathrm{e}^{-\frac{2\pi(h-\hat{l})(k-1)}{n}\mathrm{i}} C_{\hat{l},1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{h=1}^{n}\sum_{\hat{l}=1}^{n}\frac{1}{n}\mathrm{e}^{\frac{2\pi(h-1)(j-1)}{n}\mathrm{i}}\mathrm{e}^{-\frac{2\pi((h-1)+(1-\hat{l}))(k-1)}{n}\mathrm{i}}C_{\hat{l},1}\\
&= \sum_{h=1}^{n}\sum_{\hat{l}=1}^{n}\frac{1}{n}\mathrm{e}^{\frac{2\pi(h-1)((j-1)-(k-1))}{n}\mathrm{i}}\mathrm{e}^{\frac{2\pi(\hat{l}-1)(k-1)}{n}\mathrm{i}}C_{\hat{l},1}\\
&= \sum_{h=1}^{n}\sum_{\hat{l}=1}^{n}\frac{1}{\sqrt{n}}\mathrm{e}^{\frac{2\pi(h-1)(j-k)}{n}\mathrm{i}}\overline{[V_n]_{\hat{l},k}}C_{\hat{l},1}\\
&= \frac{1}{\sqrt{n}}\left(\sum_{h=1}^{n}\mathrm{e}^{\frac{2\pi(h-1)(j-k)}{n}\mathrm{i}}\right)\left(\sum_{\hat{l}=1}^{n}\overline{[V_n]_{\hat{l},k}}C_{\hat{l},1}\right)\\
&= \frac{1}{\sqrt{n}}\left(\sum_{h=1}^{n}\left(\mathrm{e}^{\frac{2\pi(j-k)}{n}\mathrm{i}}\right)^{h-1}\right)\left(\sum_{\hat{l}=1}^{n}[V_n^*]_{k,\hat{l}}I_M C_{\hat{l},1}\right)\\
&= \frac{n\delta_{j,k}}{\sqrt{n}}\sum_{\hat{l}=1}^{n}[V_n^*\otimes I_M]_{k,\hat{l}}C_{\hat{l},1} = \delta_{j,k}\sqrt{n}\sum_{\hat{l}=1}^{n}[(V_n\otimes I_M)^*]_{k,\hat{l}}C_{\hat{l},1}\\
&= \delta_{j,k}\left[\sqrt{n}(V_n\otimes I_M)^*\begin{pmatrix}C_{1,1}\\ C_{2,1}\\ \vdots\\ C_{n,1}\end{pmatrix}\right]_{k,1},
\end{aligned}
$$

where we have used the fact that if $z \neq 1$ is an nth root of unity then $\sum_{h=1}^{n} z^{h-1} = 0$ (see, e.g., [1, p. 29]).

(2)⇒(1) For every $1 \leq j,k \leq n$ we have

$$
\begin{aligned}
C_{j,k} &= [(V_n\otimes I_M)\mathrm{diag}(A_1,\ldots,A_n)(V_n\otimes I_N)^*]_{j,k}\\
&= [(V_n\otimes I_M)\mathrm{diag}(A_1,\ldots,A_n)(V_n^*\otimes I_N)]_{j,k}\\
&= \sum_{l=1}^{n}[V_n\otimes I_M]_{j,l}A_l[V_n^*\otimes I_N]_{l,k} = \sum_{l=1}^{n}[V_n]_{j,l}I_M A_l([V_n^*]_{l,k}I_N)\\
&= \sum_{l=1}^{n}[V_n]_{j,l}\overline{[V_n]_{k,l}}A_l = \sum_{l=1}^{n}\frac{1}{n}\mathrm{e}^{-\frac{2\pi(j-1)(l-1)}{n}\mathrm{i}}\mathrm{e}^{\frac{2\pi(k-1)(l-1)}{n}\mathrm{i}}A_l\\
&= \frac{1}{n}\sum_{l=1}^{n}\mathrm{e}^{\frac{2\pi(k-j)(l-1)}{n}\mathrm{i}}A_l. \qquad (5.3)
\end{aligned}
$$

Consequently, if $(j_1 - k_1) - (j_2 - k_2) = hn$ with $h \in \{-1, 0, 1\}$, then we obtain

$$\begin{aligned} C_{j_1,k_1} &= \frac{1}{n}\sum_{l=1}^{n} \mathrm{e}^{\frac{2\pi(k_1-j_1)(l-1)}{n}\mathrm{i}} A_l = \frac{1}{n}\sum_{l=1}^{n} \mathrm{e}^{\frac{2\pi(k_2-j_2-hn)(l-1)}{n}\mathrm{i}} A_l \\ &= \frac{1}{n}\sum_{l=1}^{n} \mathrm{e}^{\frac{2\pi(k_2-j_2)(l-1)}{n}\mathrm{i}} \mathrm{e}^{2\pi h(1-l)\mathrm{i}} A_l = \frac{1}{n}\sum_{l=1}^{n} \mathrm{e}^{\frac{2\pi(k_2-j_2)(l-1)}{n}\mathrm{i}} A_l \\ &= C_{j_2,k_2}. \end{aligned}$$

□

When $M = N = 1$, the previous lemma gives an eigenvalue decomposition of a circulant matrix where the eigenvector matrix is unitary. It should be mentioned that circulant matrices and tridiagonal Toeplitz matrices are practically the only types of Toeplitz matrices for which an eigenvalue decomposition is known.[1]

5.2 Sequence of Block Circulant Matrices Generated by a Continuous Function

Based on Lemma 5.1, we make the following definition.

Definition 5.2. Consider a matrix-valued function of a real variable $F\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ which is continuous and 2π-periodic. For every $n \in \mathbb{N}$ we define $C_n(F)$ as the $n \times n$ block circulant matrix with $M \times N$ blocks given by

$$\begin{aligned} C_n(F) := {} & (V_n \otimes I_M)\mathrm{diag}\left(F(0), F\left(\frac{2\pi}{n}\right), \dots, F\left(\frac{2\pi(n-1)}{n}\right)\right) \\ & (V_n \otimes I_N)^*. \end{aligned}$$

The sequence $\{C_n(F)\}$ is called the *sequence of block circulant matrices generated by* F.

---

[1] An eigenvalue decomposition of a tridiagonal Toeplitz matrix can be found, e.g., in [11, 22]. In [10] what can be found is an eigenvalue decomposition of a Hermitian tridiagonal Toeplitz matrix where the eigenvector matrix is unitary.

The next lemma provides several properties of sequences of block circulant matrices generated by continuous matrix-valued functions.

Lemma 5.2. Let $F\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ be a matrix-valued function which is continuous and 2π-periodic. Then

(1) For every $n \in \mathbb{N}$

$$[C_n(F)]_{1,k} = \frac{1}{n}\sum_{l=1}^{n} \mathrm{e}^{\frac{2\pi(k-1)(l-1)}{n}\mathrm{i}} F\left(\frac{2\pi(l-1)}{n}\right), \quad 1 \le k \le n.$$

(2) $\sigma_1(C_n(F)) \le \sigma_1(F)$ for all $n \in \mathbb{N}$.
(3) $\{C_n(F^*)\} = \{(C_n(F))^*\}$.
(4) For every matrix-valued function $G\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ which is continuous and 2π-periodic

$$\{C_n(\alpha F + \beta G)\} = \{\alpha C_n(F) + \beta C_n(G)\}, \quad \forall \alpha, \beta \in \mathbb{C}.$$

(5) For every matrix-valued function $G\colon \mathbb{R} \to \mathbb{C}^{N\times K}$ which is continuous and 2π-periodic

$$\{C_n(FG)\} = \{C_n(F)C_n(G)\},$$

where $(FG)(\omega) := F(\omega)G(\omega)$, $\omega \in \mathbb{R}$.

Proof. (1) From (5.3) this property holds.
(2) Fix $n \in \mathbb{N}$. Since the spectral norm is unitarily invariant we obtain

$$\begin{aligned}\sigma_1(C_n(F)) &= \|C_n(F)\|_2 = \|\mathrm{diag}_{1\le k\le n}(F(\omega_k))\|_2 \\ &= \max_{\substack{\mathbf{x}\in\mathbb{C}^{nN\times 1}\\ \mathbf{x}\neq 0_{nN\times 1}}} \left(\frac{\mathbf{x}^*\big(\mathrm{diag}_{1\le k\le n}(F(\omega_k))\big)^*\mathrm{diag}_{1\le k\le n}(F(\omega_k))\,\mathbf{x}}{\mathbf{x}^*\mathbf{x}}\right)^{\frac{1}{2}} \\ &= \max_{\substack{\mathbf{x}\in\mathbb{C}^{nN\times 1}\\ \mathbf{x}\neq 0_{nN\times 1}}} \left(\frac{\mathbf{x}^*\mathrm{diag}_{1\le k\le n}((F(\omega_k))^*)\,\mathrm{diag}_{1\le k\le n}(F(\omega_k))\,\mathbf{x}}{\mathbf{x}^*\mathbf{x}}\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&= \max_{\substack{\mathbf{x} \in \mathbb{C}^{nN\times 1} \\ \mathbf{x} \neq 0_{nN\times 1}}} \left(\frac{\mathbf{x}^* \mathrm{diag}_{1\leq k\leq n}\left((F(\omega_k))^* F(\omega_k) \right) \mathbf{x}}{\mathbf{x}^*\mathbf{x}} \right)^{\frac{1}{2}} \\
&= \max_{\substack{\mathbf{x} \in \mathbb{C}^{nN\times 1} \\ \mathbf{x} \neq 0_{nN\times 1}}} \left(\frac{\sum_{k=1}^{n} [\mathbf{x}^*]_{1,k} \left(F(\omega_k)\right)^* F(\omega_k) [\mathbf{x}]_{k,1}}{\mathbf{x}^*\mathbf{x}} \right)^{\frac{1}{2}} \\
&= \max_{\substack{\mathbf{x} \in \mathbb{C}^{nN\times 1} \\ \mathbf{x} \neq 0_{nN\times 1}}} \left(\frac{\sum_{k=1}^{n} [\mathbf{x}]_{k,1}^* \left(F(\omega_k)\right)^* F(\omega_k) [\mathbf{x}]_{k,1}}{\mathbf{x}^*\mathbf{x}} \right)^{\frac{1}{2}} \\
&\leq \sup_{\substack{\mathbf{x} \in \mathbb{C}^{nN\times 1} \\ \mathbf{x} \neq 0_{nN\times 1}}} \left(\frac{\sum_{k=1}^{n} \|F(\omega_k)\|_2^2 [\mathbf{x}]_{k,1}^* [\mathbf{x}]_{k,1}}{\mathbf{x}^*\mathbf{x}} \right)^{\frac{1}{2}} \\
&\leq \max_{\substack{\mathbf{x} \in \mathbb{C}^{nN\times 1} \\ \mathbf{x} \neq 0_{nN\times 1}}} \left(\frac{\sum_{k=1}^{n} \left(\max_{1\leq j\leq n} \|F(\omega_j)\|_2^2 \right) [\mathbf{x}]_{k,1}^* [\mathbf{x}]_{k,1}}{\mathbf{x}^*\mathbf{x}} \right)^{\frac{1}{2}} \\
&= \left(\max_{1\leq j\leq n} \|F(\omega_j)\|_2 \right) \max_{\substack{\mathbf{x} \in \mathbb{C}^{nN\times 1} \\ \mathbf{x} \neq 0_{nN\times 1}}} \left(\frac{\sum_{k=1}^{n} [\mathbf{x}]_{k,1}^* [\mathbf{x}]_{k,1}}{\mathbf{x}^*\mathbf{x}} \right)^{\frac{1}{2}} \\
&= \left(\max_{1\leq j\leq n} \sigma_1\left(F(\omega_j)\right) \right) \max_{\substack{\mathbf{x} \in \mathbb{C}^{nN\times 1} \\ \mathbf{x} \neq 0_{nN\times 1}}} \left(\frac{\sum_{k=1}^{n} [\mathbf{x}^*]_{1,k} [\mathbf{x}]_{k,1}}{\mathbf{x}^*\mathbf{x}} \right)^{\frac{1}{2}} \\
&= \left(\max_{1\leq j\leq n} \sigma_1\left(F(\omega_j)\right) \right) \max_{\substack{\mathbf{x} \in \mathbb{C}^{nN\times 1} \\ \mathbf{x} \neq 0_{nN\times 1}}} \left(\frac{\mathbf{x}^*\mathbf{x}}{\mathbf{x}^*\mathbf{x}} \right)^{\frac{1}{2}} \\
&= \max_{1\leq j\leq n} \sigma_1\left(F(\omega_j)\right) \leq \sigma_1(F),
\end{aligned}
$$

where $\omega_k = \frac{2\pi(k-1)}{n}$ for all $1 \leq k \leq n$.

(3) For every $n \in \mathbb{N}$ we have

$$
\begin{aligned}
(C_n(F))^* &= \left((V_n \otimes I_M) \mathrm{diag}_{1\leq k\leq n}\left(F(\omega_k)\right) (V_n \otimes I_N)^* \right)^* \\
&= (V_n \otimes I_N) \left(\mathrm{diag}_{1\leq k\leq n}\left(F(\omega_k)\right) \right)^* (V_n \otimes I_M)^* \\
&= (V_n \otimes I_N) \mathrm{diag}_{1\leq k\leq n}\left((F(\omega_k))^* \right) (V_n \otimes I_M)^* \\
&= (V_n \otimes I_N) \mathrm{diag}_{1\leq k\leq n}\left((F^*)(\omega_k) \right) (V_n \otimes I_M)^* \\
&= C_n(F^*),
\end{aligned}
$$

where $\omega_k = \frac{2\pi(k-1)}{n}$ for all $1 \le k \le n$.
(4) For every $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
&\alpha C_n(F) + \beta C_n(G) \\
&\quad = \alpha(V_n \otimes I_M)\mathrm{diag}_{1\le k\le n}\left(F\left(\omega_k\right)\right)\left(V_n \otimes I_N\right)^* \\
&\qquad + \beta(V_n \otimes I_M)\mathrm{diag}_{1\le k\le n}\left(G\left(\omega_k\right)\right)\left(V_n \otimes I_N\right)^* \\
&\quad = (V_n \otimes I_M)\mathrm{diag}_{1\le k\le n}\left(\alpha F\left(\omega_k\right)\right)\left(V_n \otimes I_N\right)^* \\
&\qquad + (V_n \otimes I_M)\mathrm{diag}_{1\le k\le n}\left(\beta G\left(\omega_k\right)\right)\left(V_n \otimes I_N\right)^* \\
&\quad = (V_n \otimes I_M)\left(\mathrm{diag}_{1\le k\le n}\left(\alpha F\left(\omega_k\right)\right) + \mathrm{diag}_{1\le k\le n}\left(\beta G\left(\omega_k\right)\right)\right) \\
&\qquad (V_n \otimes I_N)^* \\
&\quad = (V_n \otimes I_M)\mathrm{diag}_{1\le k\le n}\left(\alpha F\left(\omega_k\right) + \beta G\left(\omega_k\right)\right)\left(V_n \otimes I_N\right)^* \\
&\quad = (V_n \otimes I_M)\mathrm{diag}_{1\le k\le n}\left(\left(\alpha F + \beta G\right)\left(\omega_k\right)\right)\left(V_n \otimes I_N\right)^* \\
&\quad = C_n(\alpha F + \beta G),
\end{aligned}$$

where $\omega_k = \frac{2\pi(k-1)}{n}$ for all $1 \le k \le n$.
(5) Since $F\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ and $G\colon \mathbb{R} \to \mathbb{C}^{N\times K}$ are continuous and 2π-periodic, $FG\colon \mathbb{R} \to \mathbb{C}^{M\times K}$ is also continuous and 2π-periodic. Moreover, for every $n \in \mathbb{N}$ we have

$$\begin{aligned}
&C_n(F)C_n(G) \\
&\quad = (V_n \otimes I_M)\mathrm{diag}_{1\le k\le n}\left(F\left(\omega_k\right)\right)\left(V_n \otimes I_N\right)^*(V_n \otimes I_N) \\
&\qquad \mathrm{diag}_{1\le k\le n}\left(G\left(\omega_k\right)\right)\left(V_n \otimes I_K\right)^* \\
&\quad = (V_n \otimes I_M)\mathrm{diag}_{1\le k\le n}\left(F\left(\omega_k\right)\right)\mathrm{diag}_{1\le k\le n}\left(G\left(\omega_k\right)\right)\left(V_n \otimes I_K\right)^* \\
&\quad = (V_n \otimes I_M)\mathrm{diag}_{1\le k\le n}\left(F\left(\omega_k\right)G\left(\omega_k\right)\right)\left(V_n \otimes I_K\right)^* \\
&\quad = (V_n \otimes I_M)\mathrm{diag}_{1\le k\le n}\left(\left(FG\right)\left(\omega_k\right)\right)\left(V_n \otimes I_K\right)^* = C_n(FG),
\end{aligned}$$

where $\omega_k = \frac{2\pi(k-1)}{n}$ for all $1 \le k \le n$. □

Property (1) of Lemma 5.2 shows that our definition of sequence of block circulant matrices generated by a continuous matrix-valued function (i.e., Definition 5.2) extends the definition of sequence of circulant matrices generated by a function in the Wiener class given by Gray in [8, Equation (4.32)].

The following lemma deals with sequences of block circulant matrices generated by a continuous matrix-valued function which is Hermitian.

Lemma 5.3. Let $F\colon \mathbb{R} \to \mathbb{C}^{N\times N}$, and suppose that it is continuous, 2π-periodic and Hermitian. Then

(1) $C_n(F)$ is Hermitian for all $n \in \mathbb{N}$.
(2) For every $n \in \mathbb{N}$

$$\inf F \leq \lambda_k(C_n(F)) \leq \sup F, \qquad 1 \leq k \leq nN. \tag{5.4}$$

(3) $\{C_n(g(F))\} = \{g(C_n(F))\}$ for all $g \in C[\inf F, \sup F]$.

Proof. (1) From property (3) of Lemma 5.2 we have

$$(C_n(F))^* = C_n(F^*) = C_n(F), \quad \forall n \in \mathbb{N}.$$

(2) Fix $n \in \mathbb{N}$ and let $F(\omega) = W(\omega)\mathrm{diag}_{1\leq j\leq N}(\lambda_j(F(\omega)))\ (W(\omega))^{-1}$ be an eigenvalue decomposition of the Hermitian matrix $F(\omega)$ for all $\omega \in \mathbb{R}$. Then

$$\begin{aligned}
C_n(F) &= (V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}(F(\omega_k))(V_n \otimes I_N)^* \\
&= (V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}\Big(W(\omega_k)\mathrm{diag}_{1\leq j\leq N}\left(\lambda_j(F(\omega_k))\right)(W(\omega_k))^{-1}\Big) \\
&\quad (V_n \otimes I_N)^{-1} \\
&= (V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}(W(\omega_k))\mathrm{diag}_{1\leq k\leq n}\Big(\mathrm{diag}_{1\leq j\leq N}(\lambda_j(F(\omega_k)))\Big) \\
&\quad \mathrm{diag}_{1\leq k\leq n}\Big((W(\omega_k))^{-1}\Big)\,(V_n \otimes I_N)^{-1} \\
&= (V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}(W(\omega_k))\mathrm{diag}_{1\leq k\leq n}\Big(\mathrm{diag}_{1\leq j\leq N}(\lambda_j(F(\omega_k)))\Big) \\
&\quad \Big(\mathrm{diag}_{1\leq k\leq n}(W(\omega_k))\Big)^{-1}(V_n \otimes I_N)^{-1},
\end{aligned}$$

where $\omega_k = \frac{2\pi(k-1)}{n}$ for all $1 \leq k \leq n$. Consequently, an eigenvalue decomposition of $C_n(F)$ is

$$C_n(F) = (V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}(W(\omega_k))\mathrm{diag}_{1\leq k\leq n}\Big(\mathrm{diag}_{1\leq j\leq N}(\lambda_j(F(\omega_k)))\Big)$$
$$\big((V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}(W(\omega_k))\big)^{-1}. \tag{5.5}$$

Hence, (5.4) holds.
(3) From (5.5), we obtain

$$\begin{aligned}
&g(C_n(F))\\
&= (V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}(W(\omega_k))\mathrm{diag}_{1\leq k\leq n}\Big(\mathrm{diag}_{1\leq j\leq N}(g(\lambda_j(F(\omega_k))))\Big)\\
&\quad \Big((V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}(W(\omega_k))\Big)^{-1}\\
&= (V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}(W(\omega_k))\mathrm{diag}_{1\leq k\leq n}\Big(\mathrm{diag}_{1\leq j\leq N}(g(\lambda_j(F(\omega_k))))\Big)\\
&\quad \mathrm{diag}_{1\leq k\leq n}\Big((W(\omega_k))^{-1}\Big)(V_n \otimes I_N)^{-1}\\
&= (V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}\Big(W(\omega_k)\mathrm{diag}_{1\leq j\leq N}(g(\lambda_j(F(\omega_k))))\,(W(\omega_k))^{-1}\Big)\\
&\quad (V_n \otimes I_N)^*\\
&= (V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}(g(F(\omega_k)))(V_n \otimes I_N)^*\\
&= (V_n \otimes I_N)\mathrm{diag}_{1\leq k\leq n}((g(F))(\omega_k))(V_n \otimes I_N)^* = C_n(g(F))
\end{aligned}$$

for all $n \in \mathbb{N}$ and $g \in C[\inf F, \sup F]$. □

We finish this section with a result on sequences of block circulant matrices generated by trigonometric polynomials.

Lemma 5.4. Consider the $M \times N$ trigonometric polynomial

$$F(\omega) = \sum_{h=-m}^{m} \mathrm{e}^{h\omega\mathrm{i}}\mathsf{F}_h, \quad \forall \omega \in \mathbb{R},$$

where $m \in \mathbb{N} \cup \{0\}$. Then

$$C_n(F) = \mathrm{circ}(\mathsf{F}_0, \mathsf{F}_{-1}, \ldots, \mathsf{F}_{-m}, 0_{M\times N}, \ldots, 0_{M\times N}, \mathsf{F}_m, \ldots, \mathsf{F}_1)$$

for all $n > 2m$.

Proof. Fix $n > 2m$ and $1 \le k \le n$. From property (1) of Lemma 5.2 we have

$$\begin{aligned}
[C_n(F)]_{1,k} &= \frac{1}{n}\sum_{l=1}^{n} \mathrm{e}^{\frac{2\pi(k-1)(l-1)}{n}\mathrm{i}} F\left(\frac{2\pi(l-1)}{n}\right) \\
&= \frac{1}{n}\sum_{l=1}^{n} \mathrm{e}^{\frac{2\pi(k-1)(l-1)}{n}\mathrm{i}} \sum_{h=-m}^{m} \mathrm{e}^{\frac{2\pi h(l-1)}{n}\mathrm{i}} \mathsf{F}_h \\
&= \frac{1}{n}\sum_{l=1}^{n} \sum_{h=-m}^{m} \mathrm{e}^{\frac{2\pi(k-1)(l-1)}{n}\mathrm{i}} \mathrm{e}^{\frac{2\pi h(l-1)}{n}\mathrm{i}} \mathsf{F}_h \\
&= \frac{1}{n}\sum_{h=-m}^{m} \left(\sum_{l=1}^{n} \mathrm{e}^{\frac{2\pi(k-1)(l-1)}{n}\mathrm{i}} \mathrm{e}^{\frac{2\pi h(l-1)}{n}\mathrm{i}}\right) \mathsf{F}_h \\
&= \frac{1}{n}\sum_{h=-m}^{m} \left(\sum_{l=1}^{n} \mathrm{e}^{\frac{2\pi(h+k-1)(l-1)}{n}\mathrm{i}}\right) \mathsf{F}_h \\
&= \sum_{h=-m}^{m} \left(\frac{1}{n}\sum_{l=1}^{n} \left(\mathrm{e}^{\frac{2\pi(h+k-1)}{n}\mathrm{i}}\right)^{l-1}\right) \mathsf{F}_h \\
&= \sum_{\substack{-m \le h \le m \\ h+k-1 \in \{0,n\}}} \mathsf{F}_h \\
&= \begin{cases} \mathsf{F}_{1-k}, & \text{if } -m \le 1-k \le 0, \\ \mathsf{F}_{n+1-k}, & \text{if } 1 \le n+1-k \le m, \\ 0_{M\times N}, & \text{otherwise}, \end{cases} \\
&= \begin{cases} \mathsf{F}_{1-k}, & \text{if } 1 \le k \le m+1, \\ \mathsf{F}_{n+1-k}, & \text{if } n-m+1 \le k \le n, \\ 0_{M\times N}, & \text{otherwise}, \end{cases}
\end{aligned}$$

where we have used the fact that if $z \neq 1$ is an nth root of unity then $\sum_{l=1}^{n} z^{l-1} = 0$. □

6

Asymptotic Behaviour of Block Toeplitz Matrices

6.1 Products and Functions of Large Block Toeplitz Matrices

In this section, we use the results of the previous sections to study the asymptotic behaviour of the block Toeplitz matrices generated by a continuous matrix-valued function. The first result of the present section was given in [13] and it states that the sequence of block Toeplitz matrices and the sequence of block circulant matrices generated by the same continuous matrix-valued function are asymptotically equivalent.

Lemma 6.1. Let $F\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ be a matrix-valued function which is continuous and 2π-periodic. Then $\{T_n(F)\} \sim \{C_n(F)\}$.

Proof. From Theorem 4.3 and property (2) of Lemma 5.2, we obtain

$$\|T_n(F)\|_2, \|C_n(F)\|_2 \leq \sigma_1(F) < \infty, \quad \forall n \in \mathbb{N}.$$

To finish the proof we need to show that

$$\lim_{n\to\infty} \frac{\|T_n(F) - C_n(F)\|_{\mathrm{F}}}{\sqrt{n}} = 0.$$

Let $\{F_m\}_{m\in\mathbb{N}\cup\{0\}}$ be the sequence of partial sums of the Fourier series of F (i.e., $\{[F_m]_{r,s}\}_{m\in\mathbb{N}\cup\{0\}}$ is the sequence of partial sums of the Fourier series of $[F]_{r,s}$ for all $1 \leq r \leq M$ and $1 \leq s \leq N$):

$$F_m(\omega) = \sum_{k=-m}^{m} \mathrm{e}^{k\omega\mathrm{i}} \mathsf{F}_k, \quad m \in \mathbb{N}\cup\{0\}, \ \omega \in \mathbb{R},$$

where $\{\mathsf{F}_k\}_{k\in\mathbb{Z}}$ is the sequence of Fourier coefficients of F. Applying the Parseval theorem yields

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \|(F_m - F)(\omega)\|_F^2 d\omega \\
&= \lim_{m\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \|F_m(\omega) - F(\omega)\|_F^2 d\omega \\
&= \lim_{m\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \sum_{r=1}^{M}\sum_{s=1}^{N} |[F_m(\omega) - F(\omega)]_{r,s}|^2 d\omega \\
&= \lim_{m\to\infty} \sum_{r=1}^{M}\sum_{s=1}^{N} \frac{1}{2\pi}\int_0^{2\pi} |[F_m(\omega)]_{r,s} - [F(\omega)]_{r,s}|^2 d\omega \\
&= \sum_{r=1}^{M}\sum_{s=1}^{N} \lim_{m\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |[F_m]_{r,s}(\omega) - [F]_{r,s}(\omega)|^2 d\omega \\
&= \sum_{r=1}^{M}\sum_{s=1}^{N} 0 = 0.
\end{aligned}$$

Hence, given $\epsilon \in (0,\infty)$, there exists $m_0 \in \mathbb{N}$ such that

$$\begin{aligned}
0 &\leq \frac{1}{2\pi}\int_0^{2\pi} \|(F_{m_0} - F)(\omega)\|_F^2 d\omega \\
&= \left|\frac{1}{2\pi}\int_0^{2\pi} \|(F_{m_0} - F)(\omega)\|_F^2 d\omega\right| < \epsilon.
\end{aligned}$$

Thus,

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \|(F_{m_0} - F)(\omega)\|_F^2 d\omega\right)^{\frac{1}{2}} < \sqrt{\epsilon}, \tag{6.1}$$

and consequently, from the triangle inequality we have

$$\begin{aligned}\left|\frac{\|T_n(F)-C_n(F)\|_{\mathrm{F}}}{\sqrt{n}}\right| &= \frac{\|T_n(F)-C_n(F)\|_{\mathrm{F}}}{\sqrt{n}} \\ &< \frac{\|T_n(F)-T_n(F_{m_0})\|_{\mathrm{F}}}{\sqrt{n}} \\ &\quad + \frac{\|T_n(F_{m_0})-C_n(F_{m_0})\|_{\mathrm{F}}}{\sqrt{n}} \\ &\quad + \frac{\|C_n(F_{m_0})-C_n(F)\|_{\mathrm{F}}}{\sqrt{n}} \\ &\quad - \left(\frac{1}{2\pi}\int_0^{2\pi}\|(F_{m_0}-F)(\omega)\|_F^2 d\omega\right)^{\frac{1}{2}} \\ &\quad + \sqrt{\epsilon}, \quad \forall n \in \mathbb{N}.\end{aligned}$$

Using property (4) of Lemma 5.2 and the fact that the Frobenius norm is unitarily invariant we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{\|C_n(F_{m_0})-C_n(F)\|_{\mathrm{F}}}{\sqrt{n}} &= \left(\lim_{n\to\infty}\frac{\|C_n(F_{m_0})-C_n(F)\|_{\mathrm{F}}^2}{n}\right)^{\frac{1}{2}} \\ &= \left(\lim_{n\to\infty}\frac{\|C_n(F_{m_0}-F)\|_{\mathrm{F}}^2}{n}\right)^{\frac{1}{2}} \\ &= \left(\lim_{n\to\infty}\frac{\left\|\mathrm{diag}_{1\leq k\leq n}\left((F_{m_0}-F)\left(\frac{2\pi(k-1)}{n}\right)\right)\right\|_{\mathrm{F}}^2}{n}\right)^{\frac{1}{2}} \\ &= \left(\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\left\|(F_{m_0}-F)\left(\frac{2\pi(k-1)}{n}\right)\right\|_{\mathrm{F}}^2}{n}\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{2\pi}\int_0^{2\pi}\|(F_{m_0}-F)(\omega)\|_F^2 d\omega\right)^{\frac{1}{2}},\end{aligned}$$

where the continuity of $\|(F_{m_0}-F)(\omega)\|_F^2$ on $[0,2\pi]$ guarantees that the above limit can be written in terms of a Riemann integral. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&\frac{\|C_n(F_{m_0}) - C_n(F)\|_{\mathrm{F}}}{\sqrt{n}} - \left(\frac{1}{2\pi}\int_0^{2\pi} \|(F_{m_0} - F)(\omega)\|_F^2 d\omega\right)^{\frac{1}{2}} \\
&\quad \leq \left|\frac{\|C_n(F_{m_0}) - C_n(F)\|_{\mathrm{F}}}{\sqrt{n}} - \left(\frac{1}{2\pi}\int_0^{2\pi} \|(F_{m_0} - F)(\omega)\|_F^2 d\omega\right)^{\frac{1}{2}}\right| \\
&\quad < \sqrt{\epsilon}, \quad \forall n \geq n_0,
\end{aligned}$$

and hence,

$$\begin{aligned}
\left|\frac{\|T_n(F) - C_n(F)\|_{\mathrm{F}}}{\sqrt{n}}\right| &< \frac{\|T_n(F) - T_n(F_{m_0})\|_{\mathrm{F}}}{\sqrt{n}} \\
&\quad + \frac{\|T_n(F_{m_0}) - C_n(F_{m_0})\|_{\mathrm{F}}}{\sqrt{n}} + 2\sqrt{\epsilon}
\end{aligned}$$

for all $n \geq n_0$. From properties (2) and (3) of Lemma 4.2, and (6.1) we have

$$\begin{aligned}
\frac{\|T_n(F) - T_n(F_{m_0})\|_{\mathrm{F}}}{\sqrt{n}} &= \frac{\|T_n(F_{m_0}) - T_n(F)\|_{\mathrm{F}}}{\sqrt{n}} = \frac{\|T_n(F_{m_0} - F)\|_{\mathrm{F}}}{\sqrt{n}} \\
&\leq \left(\frac{1}{2\pi}\int_0^{2\pi} \|(F_{m_0} - F)(\omega)\|_F^2 d\omega\right)^{\frac{1}{2}} < \sqrt{\epsilon}
\end{aligned}$$

for all $n \in \mathbb{N}$, and consequently,

$$\left|\frac{\|T_n(F) - C_n(F)\|_{\mathrm{F}}}{\sqrt{n}}\right| < \frac{\|T_n(F_{m_0}) - C_n(F_{m_0})\|_{\mathrm{F}}}{\sqrt{n}} + 3\sqrt{\epsilon}$$

for all $n \geq n_0$. Applying Lemma 5.4 yields

$$\begin{aligned}
\lim_{n\to\infty} \frac{\|T_n(F_{m_0}) - C_n(F_{m_0})\|_{\mathrm{F}}}{\sqrt{n}} &= \left(\lim_{n\to\infty} \frac{\|T_n(F_{m_0}) - C_n(F_{m_0})\|_{\mathrm{F}}^2}{n}\right)^{\frac{1}{2}} \\
&= \left(\lim_{n\to\infty} \frac{\sum_{k=1}^{m_0} k(\| - \mathsf{F}_k\|_F^2 + \| - \mathsf{F}_{-k}\|_{\mathrm{F}}^2)}{n}\right)^{\frac{1}{2}} \\
&= \left(\left(\sum_{k=1}^{m_0} k(\|\mathsf{F}_k\|_{\mathrm{F}}^2 + \|\mathsf{F}_{-k}\|_{\mathrm{F}}^2)\right) \lim_{n\to\infty} \frac{1}{n}\right)^{\frac{1}{2}} = 0.
\end{aligned}$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq \frac{\|T_n(F_{m_0}) - C_n(F_{m_0})\|_{\mathrm{F}}}{\sqrt{n}} = \left| \frac{\|T_n(F_{m_0}) - C_n(F_{m_0})\|_{\mathrm{F}}}{\sqrt{n}} \right| < \sqrt{\epsilon}$$

for all $n \geq n_1$, and therefore,

$$\left| \frac{\|T_n(F) - C_n(F)\|_{\mathrm{F}}}{\sqrt{n}} \right| < 4\sqrt{\epsilon}, \quad \forall n \geq \max\{n_0, n_1\}. \qquad \square$$

We now use the previous lemma to prove an asymptotic result about the product of block Toeplitz matrices that was given in [13].

Theorem 6.2. Let $F\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ and $G\colon \mathbb{R} \to \mathbb{C}^{N\times K}$ be two matrix-valued functions which are continuous and 2π-periodic. Then $\{T_n(F)T_n(G)\} \sim \{T_n(FG)\}$.

Proof. From Lemma 6.1, we have $\{T_n(F)\} \sim \{C_n(F)\}$ and $\{T_n(G)\} \sim \{C_n(G)\}$. Hence, applying Lemma 3.2 and property (5) of Lemma 5.2 yields

$$\{T_n(F)T_n(G)\} \sim \{C_n(F)C_n(G)\} = \{C_n(FG)\}. \tag{6.2}$$

On the other hand, from Lemma 6.1 and property (1) of Lemma 3.1 we obtain

$$\{C_n(FG)\} \sim \{T_n(FG)\}. \tag{6.3}$$

Finally, from (6.2), (6.3) and property (2) of Lemma 3.1 we conclude that

$$\{T_n(F)T_n(G)\} \sim \{T_n(FG)\}. \qquad \square$$

We now use Lemma 6.1 to prove an asymptotic result about functions of Hermitian block Toeplitz matrices that was given in [12].

Theorem 6.3. Let $F\colon \mathbb{R} \to \mathbb{C}^{N\times N}$, and suppose that it is continuous, 2π-periodic and Hermitian. Then

$$\{g(T_n(F))\} \sim \{T_n(g(F))\}, \quad \forall g \in C[\inf F, \sup F].$$

Proof. From property (1) of Theorem 4.4 and property (1) of Lemma 5.3 we know that $T_n(F)$ and $C_n(F)$ are Hermitian for all $n \in \mathbb{N}$. From property (2) of Theorem 4.4 and property (2) of Lemma 5.3 we also know that

$$-\infty < \inf F \leq \lambda_k(T_n(F)), \lambda_k(C_n(F)) \leq \sup F < \infty$$

for all $1 \leq k \leq nN$ and $n \in \mathbb{N}$. Moreover, from Lemma 6.1 we have $\{T_n(F)\} \sim \{C_n(F)\}$. Consequently, applying Theorem 3.5 and property (3) of Lemma 5.3 yields

$$\{g(T_n(F))\} \sim \{g(C_n(F))\} = \{C_n(g(F))\} \tag{6.4}$$

for all $g \in C[\inf F, \sup F]$.

On the other hand, from Lemma 6.1 and property (1) of Lemma 3.1 we obtain

$$\{C_n(g(F))\} \sim \{T_n(g(F))\} \tag{6.5}$$

for all $g \in C[\inf F, \sup F]$.

Finally, from (6.4), (6.5) and property (2) of Lemma 3.1 we conclude that

$$\{g(T_n(F))\} \sim \{T_n(g(F))\}$$

for all $g \in C[\inf F, \sup F]$. □

If $0 \notin [\inf F, \sup F]$ then $g(x) = \frac{1}{x} \in C[\inf F, \sup F]$, and from Theorem 6.3 we obtain the following asymptotic result about inverses of Hermitian block Toeplitz matrices that was given in [12].

Theorem 6.4. Let $F\colon \mathbb{R} \to \mathbb{C}^{N\times N}$, and suppose that it is continuous, 2π-periodic and Hermitian. If $0 \notin [\inf F, \sup F]$ then

$$\{(T_n(F))^{-1}\} \sim \{T_n(F^{-1})\},$$

where $(F^{-1})(\omega) := (F(\omega))^{-1}$, $\omega \in \mathbb{R}$.

Three of the four asymptotic results on block Toeplitz matrices of this subsection (Lemma 6.1 and Theorems 6.2 and 6.4) were proved in [8] for Toeplitz matrices (i.e., $M = N = K = 1$) under more restrictive hypotheses, namely, generating functions in the Wiener class instead of continuous generating functions.

6.2 The Szegö Theorem for Block Toeplitz Matrices

We now proceed to prove the most famous asymptotic result on block Toeplitz matrices: the Szegö theorem for block Toeplitz matrices (see, e.g., [12]). This theorem deals with the arithmetic mean of the eigenvalues of functions of large Hermitian block Toeplitz matrices, and it has found different applications in Information Theory and Signal Processing (see, e.g., [6, 12, 13, 15, 19, 23]).

Theorem 6.5. Let $F\colon \mathbb{R} \to \mathbb{C}^{N\times N}$, and suppose that it is continuous, 2π-periodic and Hermitian. Then

$$\lim_{n\to\infty} \frac{1}{nN}\sum_{k=1}^{nN} g(\lambda_k(T_n(F))) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{N}\sum_{k=1}^{N} g(\lambda_k(F(\omega)))d\omega$$

for all $g \in C[\inf F, \sup F]$.

Proof. From Theorem 4.4, we know that the eigenvalues of the Hermitian matrix $T_n(F)$ are contained in $[\inf F, \sup F]$ for all $n \in \mathbb{N}$. Moreover, from Theorem 6.3 we have

$$\lim_{n\to\infty} \frac{\|g(T_n(F)) - T_n(g(F))\|_{\mathrm{F}}}{\sqrt{n}} = 0.$$

Consequently, since

$$\begin{aligned} 0 &\leq \left| \frac{1}{nN}\sum_{k=1}^{nN} g(\lambda_k(T_n(F))) - \frac{\mathrm{tr}(T_n(g(F)))}{nN} \right| \\ &= \left| \frac{\mathrm{tr}(g(T_n(F)))}{nN} - \frac{\mathrm{tr}(T_n(g(F)))}{nN} \right| = \left| \frac{\mathrm{tr}\left(g(T_n(F)) - T_n(g(F))\right)}{nN} \right| \\ &\leq \frac{\sqrt{nN}\|g(T_n(F)) - T_n(g(F))\|_{\mathrm{F}}}{nN} = \frac{1}{\sqrt{N}} \frac{\|g(T_n(F)) - T_n(g(F))\|_{\mathrm{F}}}{\sqrt{n}}, \end{aligned}$$

we deduce that

$$\lim_{n\to\infty} \left| \frac{1}{nN}\sum_{k=1}^{nN} g(\lambda_k(T_n(F))) - \frac{\mathrm{tr}(T_n(g(F)))}{nN} \right| = 0,$$

or equivalently,

$$\lim_{n\to\infty} \frac{1}{nN} \sum_{k=1}^{nN} g(\lambda_k(T_n(F))) - \frac{\mathrm{tr}(T_n(g(F)))}{nN} = 0.$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{nN} \sum_{k=1}^{nN} g(\lambda_k(T_n(F))) &= \lim_{n\to\infty} \frac{\mathrm{tr}(T_n(g(F)))}{nN} \\
&= \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mathrm{tr}([T_n(g(F))]_{k,k})}{nN} \\
&= \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mathrm{tr}\left(\frac{1}{2\pi}\int_0^{2\pi} g(F(\omega))d\omega\right)}{nN} \\
&= \lim_{n\to\infty} \frac{n\mathrm{tr}\left(\frac{1}{2\pi}\int_0^{2\pi} g(F(\omega))d\omega\right)}{nN} \\
&= \frac{\mathrm{tr}\left(\frac{1}{2\pi}\int_0^{2\pi} g(F(\omega))d\omega\right)}{N} \\
&= \frac{1}{2\pi N}\mathrm{tr}\left(\int_0^{2\pi} g(F(\omega))d\omega\right) \\
&= \frac{1}{2\pi N}\sum_{r=1}^{N}\left[\int_0^{2\pi} g(F(\omega))d\omega\right]_{r,r} \\
&= \frac{1}{2\pi N}\sum_{r=1}^{N}\int_0^{2\pi} [g(F(\omega))]_{r,r}\, d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{N}\sum_{r=1}^{N} [g(F(\omega))]_{r,r}\, d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{N}\mathrm{tr}\left(g(F(\omega))\right) d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{N}\sum_{k=1}^{N} g(\lambda_k(F(\omega)))d\omega. \qquad \square
\end{aligned}$$

Although in the present monograph we have only used the Riemann integral, it should be mentioned that the Szegö theorem for block Toeplitz matrices is not only known for continuous generating functions, but also for generating functions whose entries are Lebesgue integrable functions [21].

We now prove that the Szegö theorem for block Toeplitz matrices is also true if the sequence of Hermitian block Toeplitz matrices $\{T_n(F)\}$ is replaced by any sequence of Hermitian matrices $\{A_n\}$ such that $\{A_n\} \sim \{T_n(F)\}$, and $[\inf F, \sup F]$ is replaced by a bigger closed interval containing the eigenvalues of A_n for all $n \in \mathbb{N}$. Theorems 6.2 and 6.3 are very useful results to obtain this kind of matrix sequence $\{A_n\}$.

Theorem 6.6. Let A_n be an $nN \times nN$ Hermitian matrix for all $n \in \mathbb{N}$. Suppose that $\{A_n\} \sim \{T_n(F)\}$, where $F\colon \mathbb{R} \to \mathbb{C}^{N \times N}$ is continuous, 2π-periodic and Hermitian. Consider $a, b \in \mathbb{R}$ satisfying $[\inf F, \sup F] \subseteq [a, b]$ and $a \leq \lambda_k(A_n) \leq b$ for all $1 \leq k \leq nN$ and $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{1}{nN} \sum_{k=1}^{nN} g(\lambda_k(A_n)) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{N} \sum_{k=1}^{N} g(\lambda_k(F(\omega)))d\omega$$

for all $g \in C[a, b]$.

Proof. For every $n \in \mathbb{N}$, from Theorem 4.4 we know that $T_n(F)$ is Hermitian and

$$\lambda_k(T_n(F)) \in [\inf F, \sup F] \subseteq [a, b], \quad 1 \leq k \leq nN.$$

Hence, from Theorems 3.6 and 6.5 we obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{nN} \sum_{k=1}^{nN} g(\lambda_k(A_n)) &= \frac{1}{N} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{nN} g(\lambda_k(A_n)) \\
&= \frac{1}{N} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{nN} g(\lambda_k(T_n(F)))
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{1}{nN} \sum_{k=1}^{nN} g(\lambda_k(T_n(F)))$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{N} \sum_{k=1}^{N} g(\lambda_k(F(\omega)))d\omega. \qquad \square$$

Unlike in [3, 5, 14], the asymptotic results that we have presented in this monograph do not provide any information about a concrete entry of a large matrix of the type considered.

7

Applications to Vector Random Processes

7.1 Vector Asymptotically WSS Processes

We begin this section by extending the definition of asymptotically WSS (AWSS) process introduced in [8] to vector random processes.

Definition 7.1. An N-dimensional vector random process $\{\mathbf{x}_n : n \in \mathbb{Z}\}$ is said to be *AWSS* if it has constant mean and

$$\{\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)\} \sim \{T_n(X)\}$$

for some matrix-valued function $X : \mathbb{R} \to \mathbb{C}^{N \times N}$ which is continuous and 2π-periodic.

Let $\{\mathbf{x}_n : n \in \mathbb{Z}\}$ be a vector WSS process with continuous power spectral density X. From Theorem 4.3 $\{\|T_n(X)\|_2\} = \{\sigma_1(T_n(X))\}$ is bounded. Hence, applying property (1) of Lemma 3.3 yields $\{\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)\} = \{T_n(X)\} \sim \{T_n(X)\}$. Consequently, a vector WSS process with continuous power spectral density is AWSS.

The next result states that when a vector zero mean AWSS process passes through a discrete-time causal FIR MIMO filter, the filter output is also a vector zero mean AWSS process.

Theorem 7.1. Consider $m \in \mathbb{N} \cup \{0\}$ and a matrix-valued function $X : \mathbb{R} \to \mathbb{C}^{N \times N}$ which is continuous and 2π-periodic. Let $\{\mathbf{x}_{n-m} : n \in \mathbb{Z}\}$ be an N-dimensional vector AWSS process with zero mean (i.e., $\mathrm{E}(\mathbf{x}_{n-m}) = 0_{N\times 1}$ for all $n \in \mathbb{Z}$) and $\{\mathrm{E}(\mathbf{x}_{n-m:1-m}\mathbf{x}^*_{n-m:1-m})\} \sim \{T_n(X)\}$. Suppose that $\{\mathbf{y}_n : n \in \mathbb{Z}\}$ is the M-dimensional vector random process given by

$$\mathbf{y}_n = \sum_{k=0}^{m} \mathsf{H}_{-k}\mathbf{x}_{n-k}, \quad \forall n \in \mathbb{Z},$$

where $\mathsf{H}_{-k} \in \mathbb{C}^{M\times N}$ with $0 \leq k \leq m$. Then $\{\mathbf{y}_n : n \in \mathbb{Z}\}$ is AWSS with zero mean and

$$\{\mathrm{E}(\mathbf{y}_{n:1}\mathbf{y}^*_{n:1})\} \sim \{T_n(HXH^*)\},$$

where $H(\omega) = \sum_{k=-m}^{0} \mathrm{e}^{k\omega\mathrm{i}}\mathsf{H}_k$ for all $\omega \in \mathbb{R}$.

Proof. $\{\mathbf{y}_n : n \in \mathbb{Z}\}$ has zero mean because

$$\begin{aligned}\mathrm{E}(\mathbf{y}_n) &= \mathrm{E}\left(\sum_{k=0}^{m} \mathsf{H}_{-k}\mathbf{x}_{n-k}\right)\\ &= \sum_{k=0}^{m} \mathsf{H}_{-k}\mathrm{E}(\mathbf{x}_{n-k}) = \sum_{k=0}^{m} \mathsf{H}_{-k}0_{N\times 1} = 0_{M\times 1}, \quad \forall n \in \mathbb{Z}.\end{aligned}$$

For every $n \in \mathbb{N}$ we have

$$\mathbf{y}_{n:1} = H_n\mathbf{x}_{n:1-m},$$

where H_n is the $n \times (n+m)$ block Toeplitz matrix with $M \times N$ blocks given by (4.4). Therefore,

$$\begin{aligned}\{\mathrm{E}(\mathbf{y}_{n:1}\mathbf{y}^*_{n:1})\} &= \{\mathrm{E}(H_n\mathbf{x}_{n:1-m}(H_n\mathbf{x}_{n:1-m})^*)\}\\ &= \{\mathrm{E}(H_n\mathbf{x}_{n:1-m}\mathbf{x}^*_{n:1-m}H_n^*)\}\\ &= \{H_n\mathrm{E}(\mathbf{x}_{n:1-m}\mathbf{x}^*_{n:1-m})H_n^*\}.\end{aligned}$$

Since

$$\|H_n - (T_n(H)|0_{nM\times mN})\|_{\mathrm{F}}^2 = \sum_{k=1}^{m} k\|\mathsf{H}_{-k}\|_{\mathrm{F}}^2, \quad \forall n \geq m,$$

from property (2) of Lemma 3.3 we obtain

$$\{H_n - (T_n(H)|0_{nM\times mN})\} \sim \{0_{nM\times(n+m)N}\}. \tag{7.1}$$

Fix $n \in \mathbb{N}$, if $T_n(H) = U\Sigma V^*$ is a singular value decomposition of $T_n(H)$ then

$$(T_n(H)|0_{nM\times mN}) = U(\Sigma|0_{nM\times mN}) \begin{pmatrix} V & 0_{nN\times mN} \\ 0_{mN\times nN} & I_{mN} \end{pmatrix}^*$$

is a singular value decomposition of $(T_n(H)|0_{nM\times mN})$. Hence, from Theorem 4.3 $\{\|(T_n(H)|0_{nM\times mN})\|_2\} = \{\sigma_1((T_n(H)|0_{nM\times mN}))\} = \{\sigma_1(T_n(H))\} = \{\|T_n(H)\|_2\}$ is bounded. Consequently, applying property (1) of Lemma 3.3 yields

$$\{T_n(H)\} \sim \{T_n(H)\}, \tag{7.2}$$

and

$$\{(T_n(H)|0_{nM\times mN})\} \sim \{(T_n(H)|0_{nM\times mN})\}. \tag{7.3}$$

From (7.1), (7.3) and property (4) of Lemma 3.1 we have

$$\{H_n\} \sim \{(T_n(H)|0_{nM\times mN})\}. \tag{7.4}$$

Applying property (7) of Lemma 3.1 and property (1) of Lemma 4.2 yields

$$\{H_n^*\} \sim \left\{ \left(\frac{(T_n(H))^*}{0_{mN\times nM}} \right) \right\} = \left\{ \left(\frac{T_n(H^*)}{0_{mN\times nM}} \right) \right\}. \tag{7.5}$$

Since $\{\mathrm{E}(\mathbf{x}_{n:1-m}\mathbf{x}_{n:1-m}^*)\} \sim \{T_{n+m}(X)\}$, from (7.4) and Lemma 3.2 we obtain

$$\{H_n \mathrm{E}(\mathbf{x}_{n:1-m}\mathbf{x}_{n:1-m}^*)\} \sim \{(T_n(H)|0_{nM\times mN})T_{n+m}(X)\}. \tag{7.6}$$

Using (7.5), (7.6) and Lemma 3.2 yields

$$\begin{aligned} \{\mathrm{E}(\mathbf{y}_{n:1}\mathbf{y}_{n:1}^*)\} &= \{H_n \mathrm{E}(\mathbf{x}_{n:1-m}\mathbf{x}_{n:1-m}^*)H_n^*\} \\ &\sim \left\{ (T_n(H)|0_{nM\times mN})T_{n+m}(X) \left(\frac{T_n(H^*)}{0_{mN\times nM}} \right) \right\} \\ &= \{T_n(H)T_n(X)T_n(H^*)\}. \end{aligned} \tag{7.7}$$

From Theorem 6.2, we have

$$\{T_n(X)T_n(H^*)\} \sim \{T_n(XH^*)\}, \tag{7.8}$$

and

$$\{T_n(H)T_n(XH^*)\} \sim \{T_n(HXH^*)\}. \tag{7.9}$$

Using (7.2), (7.8) and Lemma 3.2 yields

$$\{T_n(H)T_n(X)T_n(H^*)\} \sim \{T_n(H)T_n(XH^*)\}. \tag{7.10}$$

From (7.9), (7.10) and property (2) of Lemma 3.1 we obtain

$$\{T_n(H)T_n(X)T_n(H^*)\} \sim \{T_n(HXH^*)\}. \tag{7.11}$$

Applying (7.7), (7.11) and property (2) of Lemma 3.1 yields

$$\{\mathrm{E}(\mathbf{y}_{n:1}\mathbf{y}_{n:1}^*)\} \sim \{T_n(HXH^*)\}. \qquad \square$$

The following theorem shows that when a vector zero mean AWSS process that starts at some fixed time, passes through a discrete-time causal infinite impulse response (IIR) MIMO filter, the filter output is also a vector zero mean AWSS process.

Theorem 7.2. Consider a matrix-valued function $X:\mathbb{R}\to\mathbb{C}^{N\times N}$ which is continuous and 2π-periodic. Let $\{\mathbf{x}_n : n\in\mathbb{Z}\}$ be an N-dimensional vector zero mean AWSS process with $\{\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)\} \sim \{T_n(X)\}$ and $\mathbf{x}_n = 0_{N\times 1}$ for all $n\leq 0$. Suppose that $\{\mathbf{y}_n : n\in\mathbb{Z}\}$ is the M-dimensional vector random process given by

$$\mathbf{y}_n = \sum_{k=0}^{\infty} \mathsf{H}_{-k}\mathbf{x}_{n-k} := \left(\sum_{k=0}^{\infty}\left[\mathsf{H}_{-k}\mathbf{x}_{n-k}\right]_{r,1}\right)_{1\leq r\leq M,1}, \quad \forall n\in\mathbb{Z},$$

where $\{\mathsf{H}_k\}_{k\in\mathbb{Z}}$, with $\mathsf{H}_k = 0_{M\times N}$ for all $k\in\mathbb{N}$, is the sequence of Fourier coefficients of a matrix-valued function $H:\mathbb{R}\to\mathbb{C}^{M\times N}$ which is continuous and 2π-periodic. Then $\{\mathbf{y}_n : n\in\mathbb{Z}\}$ is AWSS with zero mean and

$$\{\mathrm{E}(\mathbf{y}_{n:1}\mathbf{y}_{n:1}^*)\} \sim \{T_n(HXH^*)\}.$$

Proof. For every $n \in \mathbb{Z}$ we have

$$\mathbf{y}_n = \begin{cases} \sum_{k=0}^{n-1} \mathsf{H}_{-k}\mathbf{x}_{n-k}, & \text{if } n \in \mathbb{N}, \\ 0_{M\times 1}, & \text{if } n \leq 0. \end{cases} \tag{7.12}$$

Hence,

$$\begin{aligned} \mathrm{E}(\mathbf{y}_n) &= \begin{cases} \mathrm{E}\left(\sum_{k=0}^{n-1} \mathsf{H}_{-k}\mathbf{x}_{n-k}\right) = \sum_{k=0}^{n-1} \mathsf{H}_{-k}\mathrm{E}(\mathbf{x}_{n-k}), & \text{if } n \in \mathbb{N}, \\ \mathrm{E}(0_{M\times 1}) = 0_{M\times 1}, & \text{if } n \leq 0, \end{cases} \\ &= \begin{cases} \sum_{k=0}^{n-1} \mathsf{H}_{-k}0_{N\times 1} = 0_{M\times 1}, & \text{if } n \in \mathbb{N}, \\ 0_{M\times 1}, & \text{if } n \leq 0, \end{cases} \\ &= 0_{M\times 1} \end{aligned}$$

for all $n \in \mathbb{Z}$. Thus, $\{\mathbf{y}_n : n \in \mathbb{Z}\}$ has zero mean.

From (7.12) we obtain

$$\mathbf{y}_{n:1} = T_n(H)\,\mathbf{x}_{n:1}, \quad \forall n \in \mathbb{N}.$$

Therefore,

$$\begin{aligned} \{\mathrm{E}(\mathbf{y}_{n:1}\mathbf{y}_{n:1}^*)\} &= \{\mathrm{E}(T_n(H)\mathbf{x}_{n:1}(T_n(H)\mathbf{x}_{n:1})^*)\} \\ &= \{\mathrm{E}(T_n(H)\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*(T_n(H))^*)\} \\ &= \{T_n(H)\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)(T_n(H))^*\}. \end{aligned}$$

From Theorem 4.3, $\{\|(T_n(H))^*\|_2\} = \{\|T_n(H)\|_2\} = \{\sigma_1(T_n(H))\}$ is bounded. Consequently, applying property (1) of Lemma 3.3 yields

$$\{(T_n(H))^*\} \sim \{(T_n(H))^*\}, \tag{7.13}$$

and

$$\{T_n(H)\} \sim \{T_n(H)\}. \tag{7.14}$$

Using (7.13) and property (1) of Lemma 4.2 gives

$$\{(T_n(H))^*\} \sim \{(T_n(H))^*\} = \{T_n(H^*)\}. \tag{7.15}$$

Since $\{\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)\} \sim \{T_n(X)\}$, from (7.14) and Lemma 3.2 we obtain

$$\{T_n(H)\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)\} \sim \{T_n(H)T_n(X)\}. \tag{7.16}$$

Applying (7.15), (7.16) and Lemma 3.2 yields

$$\begin{aligned}\{\mathrm{E}(\mathbf{y}_{n:1}\mathbf{y}_{n:1}^*)\} &= \{T_n(H)\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)(T_n(H))^*\} \\ &\sim \{T_n(H)T_n(X)T_n(H^*)\}. \end{aligned} \tag{7.17}$$

From Theorem 6.2, we have

$$\{T_n(X)T_n(H^*)\} \sim \{T_n(XH^*)\}, \tag{7.18}$$

and

$$\{T_n(H)T_n(XH^*)\} \sim \{T_n(HXH^*)\}. \tag{7.19}$$

Using (7.14), (7.18) and Lemma 3.2 yields

$$\{T_n(H)T_n(X)T_n(H^*)\} \sim \{T_n(H)T_n(XH^*)\}. \tag{7.20}$$

From (7.19), (7.20) and property (2) of Lemma 3.1 we obtain

$$\{T_n(H)T_n(X)T_n(H^*)\} \sim \{T_n(HXH^*)\}. \tag{7.21}$$

Applying (7.17), (7.21) and property (2) of Lemma 3.1 yields

$$\{\mathrm{E}(\mathbf{y}_{n:1}\mathbf{y}_{n:1}^*)\} \sim \{T_n(HXH^*)\}. \qquad \square$$

Theorem 7.2 is a generalization of [13, Theorem 6], which is a result that gives sufficient conditions for vector moving average (MA) processes to be AWSS. It should be mentioned that in [13] we also provided sufficient conditions for vector autoregressive (AR) processes and vector autoregressive moving average (ARMA) processes to be AWSS.

7.2 Differential Entropy Rate of Certain Vector AWSS Processes

We finish this section by computing the differential entropy rate of certain vector AWSS processes.

Let $X\colon \mathbb{R} \to \mathbb{C}^{N\times N}$ be a matrix-valued function, and suppose that it is continuous, 2π-periodic and Hermitian. Consider an N-dimensional vector random process $\{\mathbf{x}_n : n \in \mathbb{Z}\}$ which is proper, Gaussian with zero mean and AWSS with $\{\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)\} \sim \{T_n(X)\}$.

Since the correlation matrix of a random vector is Hermitian, $\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)$ is diagonalizable for all $n \in \mathbb{N}$. Hence, $\det(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)) = \prod_{k=1}^{nN} \lambda_k(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*))$ for all $n \in \mathbb{N}$. Consider $a, b \in \mathbb{R}$ satisfying $[\inf X, \sup X] \subseteq [a, b]$ and $a \leq \lambda_k(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)) \leq b$ for all $1 \leq k \leq nN$ and $n \in \mathbb{N}$. Suppose that $a > 0$, then

$$\det(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)) = \prod_{k=1}^{nN} \lambda_k(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)) \geq a^{nN} > 0, \quad \forall n \in \mathbb{N}.$$

As $\det(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)) \neq 0$ and the random vector $\mathbf{x}_{n:1}$ is proper and Gaussian with zero mean, from [18, Theorem 2] the differential entropy $h(\mathbf{x}_{n:1})$ of $\mathbf{x}_{n:1}$ is given by

$$\begin{aligned} h(\mathbf{x}_{n:1}) &= \log_2\left((\pi\mathrm{e})^{nN} \det\left(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)\right)\right) \\ &= \log_2\left((\pi\mathrm{e})^{nN}\right) + \log_2 \det\left(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)\right) \\ &= nN \log_2(\pi\mathrm{e}) + \log_2 \prod_{k=1}^{nN} \lambda_k(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)) \\ &= nN \log_2(\pi\mathrm{e}) + \sum_{k=1}^{nN} \log_2 \lambda_k(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)), \quad \forall n \in \mathbb{N}. \end{aligned}$$

Therefore, since the differential entropy rate of $\{\mathbf{x}_n \colon n \in \mathbb{Z}\}$ is defined as $\lim_{n\to\infty} \frac{1}{n} h(\mathbf{x}_{n:1})$, from Theorem 6.6 we obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} h(\mathbf{x}_{n:1}) &= \lim_{n\to\infty} N \log_2(\pi\mathrm{e}) + \frac{1}{n} \sum_{k=1}^{nN} \log_2 \lambda_k(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)) \\ &= N \log_2(\pi\mathrm{e}) + \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{nN} \log_2 \lambda_k(\mathrm{E}(\mathbf{x}_{n:1}\mathbf{x}_{n:1}^*)) \\ &= N \log_2(\pi\mathrm{e}) + \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=1}^{N} \log_2 \lambda_k(X(\omega)) d\omega \\ &= N \log_2(\pi\mathrm{e}) + \frac{1}{2\pi} \int_0^{2\pi} \log_2 \prod_{k=1}^{N} \lambda_k(X(\omega)) d\omega \\ &= N \log_2(\pi\mathrm{e}) + \frac{1}{2\pi} \int_0^{2\pi} \log_2 \det(X(\omega)) d\omega. \end{aligned}$$

A

Asymptotically 2-Equivalent Sequences of Matrices

In Definition 3.1, we extended the Gray concept of asymptotically equivalent sequences of matrices to sequences of non-square matrices. We present here another definition that is also an extension to sequences of non-square matrices of that concept of Gray.

Definition A.1. Consider two strictly increasing sequences of natural numbers $\{d_n^{(1)}\}$ and $\{d_n^{(2)}\}$. Let A_n and B_n be $d_n^{(1)} \times d_n^{(2)}$ matrices for all $n \in \mathbb{N}$. We say that the sequences $\{A_n\}$ and $\{B_n\}$ are *asymptotically 2-equivalent*, and write $\{A_n\} \sim_2 \{B_n\}$, if they satisfy (3.1) and

$$\lim_{n\to\infty} \frac{\|A_n - B_n\|_{\mathrm{F}}}{\sqrt{\max\left\{d_n^{(1)}, d_n^{(2)}\right\}}} = 0.$$

Definition A.1 was introduced in [12, Appendix] for the case in which $\{d_n^{(1)}\} = \{d_n^{(2)}\}$.

The following lemma relates Definition A.1 with Definition 3.1.

Lemma A.1. Consider two strictly increasing sequences of natural numbers $\{d_n^{(1)}\}$ and $\{d_n^{(2)}\}$. Let A_n and B_n be $d_n^{(1)} \times d_n^{(2)}$ matrices for all $n \in \mathbb{N}$. If $\{A_n\} \sim \{B_n\}$ then $\{A_n\} \sim_2 \{B_n\}$.

Proof. Since $\{d_n^{(1)}\}$ and $\{d_n^{(2)}\}$ are strictly increasing sequences of natural numbers, they satisfy $d_n^{(1)}, d_n^{(2)} \geq n$ for all $n \in \mathbb{N}$. Consequently,

$$0 \leq \frac{\|A_n - B_n\|_F}{\sqrt{\max\left\{d_n^{(1)}, d_n^{(2)}\right\}}} \leq \frac{\|A_n - B_n\|_F}{\sqrt{n}} \to 0. \qquad \square$$

As we will now prove Lemma 3.2 is not true when $\sim$ is replaced by $\sim_2$, and this is the reason why we have used $\sim$ instead of $\sim_2$ in the present monograph. Observe that if $\{A_n\} = \{(I_n|0_{n\times(n^2-n)})\}$, $\{B_n\} = \{0_{n\times n^2}\}$, $\{C_n\} = \{A_n^\top\}$, and $\{D_n\} = \{B_n^\top\}$, then

$$\{\|C_n\|_2\} = \left\{\left\|A_n^\top\right\|_2\right\} = \{\|A_n\|_2\} = \{1\}$$

and

$$\{\|D_n\|_2\} = \left\{\left\|B_n^\top\right\|_2\right\} = \{\|B_n\|_2\} = \{0\}.$$

Moreover,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\|C_n - D_n\|_\mathrm{F}}{\sqrt{n^2}} &= \lim_{n\to\infty} \frac{\left\|A_n^\top - B_n^\top\right\|_\mathrm{F}}{\sqrt{n^2}} = \lim_{n\to\infty} \frac{\left\|(A_n - B_n)^\top\right\|_\mathrm{F}}{\sqrt{n^2}} \\
&= \lim_{n\to\infty} \frac{\|A_n - B_n\|_\mathrm{F}}{\sqrt{n^2}} = \lim_{n\to\infty} \frac{\left\|\left(I_n|0_{n\times(n^2-n)}\right)\right\|_\mathrm{F}}{\sqrt{n^2}} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \frac{\|A_nC_n - B_nD_n\|_F}{\sqrt{n}} &= \lim_{n\to\infty} \frac{\|I_n - 0_{n\times n}\|_F}{\sqrt{n}} \\
&= \lim_{n\to\infty} \frac{\|I_n\|_F}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} = 1 \neq 0.
\end{aligned}$$

Therefore, $\{A_n\} \sim_2 \{B_n\}$ and $\{C_n\} \sim_2 \{D_n\}$, but the two sequences of matrices $\{A_nC_n\}$ and $\{B_nD_n\}$ are not asymptotically 2-equivalent.

B

The Wiener Class

We present here the Wiener class in detail.

Definition B.1. A function in *the Wiener class* is a function of the form

$$F(\omega) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{k\omega\mathrm{i}} \mathsf{A}_k := \left(\lim_{m\to\infty} \sum_{k=-m}^{m} \mathrm{e}^{k\omega\mathrm{i}} [\mathsf{A}_k]_{r,s} \right)_{1\leq r\leq M, 1\leq s\leq N}, \quad \text{(B.1)}$$

where $\omega \in \mathbb{R}$, $\mathsf{A}_k \in \mathbb{C}^{M\times N}$ with $k \in \mathbb{Z}$, and

$$\sum_{k=-\infty}^{\infty} |[\mathsf{A}_k]_{r,s}| < \infty$$

for all $1 \leq r \leq M$ and $1 \leq s \leq N$.

If $1 \leq r \leq M$ and $1 \leq s \leq N$, $\{\sum_{k=-m}^{m} |[\mathsf{A}_k]_{r,s}|\}$ is a Cauchy sequence because it is convergent. Therefore, for every $\epsilon \in (0,\infty)$ there exists $m_0 \in \mathbb{N}$ such that

$$\left| \sum_{k=-m}^{m} |[\mathsf{A}_k]_{r,s}| - \sum_{k=-n}^{n} |[\mathsf{A}_k]_{r,s}| \right| < \epsilon$$

for all $m, n \geq m_0$. Since

$$\left|\sum_{k=-m}^{m} e^{k\omega i}[\mathsf{A}_k]_{r,s} - \sum_{k=-n}^{n} e^{k\omega i}[\mathsf{A}_k]_{r,s}\right|$$

$$= \left|\sum_{k=-\max\{m,n\}}^{-\min\{m,n\}} e^{k\omega i}[\mathsf{A}_k]_{r,s} + \sum_{k=\min\{m,n\}}^{\max\{m,n\}} e^{k\omega i}[\mathsf{A}_k]_{r,s}\right|$$

$$\leq \sum_{k=-\max\{m,n\}}^{-\min\{m,n\}} \left|e^{k\omega i}[\mathsf{A}_k]_{r,s}\right| + \sum_{k=\min\{m,n\}}^{\max\{m,n\}} \left|e^{k\omega i}[\mathsf{A}_k]_{r,s}\right|$$

$$= \sum_{k=-\max\{m,n\}}^{-\min\{m,n\}} |[\mathsf{A}_k]_{r,s}| + \sum_{k=\min\{m,n\}}^{\max\{m,n\}} |[\mathsf{A}_k]_{r,s}|$$

$$= \left|\sum_{k=-\max\{m,n\}}^{-\min\{m,n\}} |[\mathsf{A}_k]_{r,s}| + \sum_{k=\min\{m,n\}}^{\max\{m,n\}} |[\mathsf{A}_k]_{r,s}|\right|$$

$$= \left|\sum_{k=-m}^{m} |[\mathsf{A}_k]_{r,s}| - \sum_{k=-n}^{n} |[\mathsf{A}_k]_{r,s}|\right| < \epsilon,$$

for all $m, n \geq m_0$ and $\omega \in \mathbb{R}$, from the Cauchy criterion for uniform convergence (see, e.g., [20, p. 147]) we obtain that the sequence of functions $\{\sum_{k=-m}^{m} e^{k\omega i}[\mathsf{A}_k]_{r,s}\}$ converges uniformly on $\mathbb{R}$. Hence, $\{\sum_{k=-m}^{m} e^{k\omega i}[\mathsf{A}_k]_{r,s}\}$ converges pointwise on $\mathbb{R}$ to $[F]_{r,s}$. Thus, F is well defined and $F(\omega) \in \mathbb{C}^{M\times N}$ for all $\omega \in \mathbb{R}$.

The following two lemmas provide several fundamental properties of the functions in the Wiener class.

Lemma B.1. Let $F\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ be the function given by (B.1). Then,

(1) F is continuous.
(2) F is 2π-periodic.
(3) $\{\mathsf{A}_k\}_{k\in\mathbb{Z}}$ is the sequence of Fourier coefficients of F.

Proof. (1) If $1 \le r \le M$ and $1 \le s \le N$, $[F]_{r,s}$ is continuous on $\mathbb{R}$ because the sequence of continuous functions $\{\sum_{k=-m}^{m} \mathrm{e}^{k\omega \mathrm{i}}[\mathsf{A}_k]_{r,s}\}$ converges uniformly on $\mathbb{R}$ to $[F]_{r,s}$.
(2) If $1 \le r \le M$ and $1 \le s \le N$ then $[F]_{r,s}$ is 2π-periodic since

$$\begin{aligned}[F]_{r,s}(\omega + 2\pi) &= \lim_{m\to\infty} \sum_{k=-m}^{m} \mathrm{e}^{k(\omega+2\pi)\mathrm{i}}[\mathsf{A}_k]_{r,s} \\ &= \lim_{m\to\infty} \sum_{k=-m}^{m} \mathrm{e}^{k\omega\mathrm{i}}[\mathsf{A}_k]_{r,s} = [F]_{r,s}(\omega), \quad \forall \omega \in \mathbb{R}.\end{aligned}$$

(3) Fix $1 \le r \le M$ and $1 \le s \le N$. Since $\{\sum_{k=-m}^{m} \mathrm{e}^{k\omega \mathrm{i}}[\mathsf{A}_k]_{r,s}\}$ converges uniformly on $\mathbb{R}$ to $[F]_{r,s}$, for every $\epsilon \in (0, \infty)$ there exists $m_0 \in \mathbb{N}$ such that

$$\left| \sum_{k=-m}^{m} \mathrm{e}^{k\omega\mathrm{i}}[\mathsf{A}_k]_{r,s} - [F]_{r,s}(\omega) \right| < \epsilon$$

for all $m \ge m_0$ and $\omega \in \mathbb{R}$. Consequently,

$$\begin{aligned}&\left| \mathrm{e}^{-h\omega\mathrm{i}} \sum_{k=-m}^{m} \mathrm{e}^{k\omega\mathrm{i}}[\mathsf{A}_k]_{r,s} - \mathrm{e}^{-h\omega\mathrm{i}}[F]_{r,s}(\omega) \right| \\ &\qquad = \left|\mathrm{e}^{-h\omega\mathrm{i}}\right| \left| \sum_{k=-m}^{m} \mathrm{e}^{k\omega\mathrm{i}}[\mathsf{A}_k]_{r,s} - [F]_{r,s}(\omega) \right| \\ &\qquad = \left| \sum_{k=-m}^{m} \mathrm{e}^{k\omega\mathrm{i}}[\mathsf{A}_k]_{r,s} - [F]_{r,s}(\omega) \right| < \epsilon,\end{aligned}$$

for all $m \ge m_0$, $\omega \in \mathbb{R}$ and $h \in \mathbb{Z}$. Thus, $\{\mathrm{e}^{-h\omega\mathrm{i}} \sum_{k=-m}^{m} \mathrm{e}^{k\omega \mathrm{i}}[\mathsf{A}_k]_{r,s}\}$ converges uniformly on $\mathbb{R}$ to $\mathrm{e}^{-h\omega\mathrm{i}}[F]_{r,s}(\omega)$ for all $h \in \mathbb{Z}$. Hence, applying a well known result on uniform convergence and integration (see, e.g., [20, Theorem 7.16]) yields

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-h\omega\mathrm{i}}[F]_{r,s}(\omega) d\omega &= \frac{1}{2\pi} \lim_{m\to\infty} \int_0^{2\pi} \mathrm{e}^{-h\omega\mathrm{i}} \sum_{k=-m}^{m} \mathrm{e}^{k\omega\mathrm{i}}[\mathsf{A}_k]_{r,s} d\omega \\ &= \lim_{m\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=-m}^{m} \mathrm{e}^{(k-h)\omega\mathrm{i}}[\mathsf{A}_k]_{r,s} d\omega\end{aligned}$$

$$= \lim_{m\to\infty} \sum_{k=-m}^{m} [\mathsf{A}_k]_{r,s} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{(k-h)\omega \mathrm{i}} d\omega$$

$$= \lim_{m\to\infty} \sum_{k=-m}^{m} [\mathsf{A}_k]_{r,s} \delta_{k,h} = [\mathsf{A}_h]_{r,s} \quad \forall h \in \mathbb{Z},$$

where $\delta_{k,h}$ is the Kronecker delta. □

Lemma B.2. Let $\{\mathsf{F}_k\}_{k\in\mathbb{Z}}$ be the sequence of Fourier coefficients of a matrix-valued function $F\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ which is continuous and 2π-periodic. F is in the Wiener class if and only if

$$\sum_{k=-\infty}^{\infty} |[\mathsf{F}_k]_{r,s}| < \infty, \quad 1 \le r \le M, \; 1 \le s \le N. \tag{B.2}$$

Proof. From Lemma B.1, if F is in the Wiener class then (B.2) holds. We now proceed to show that the reciprocal is also true. We consider the function in the Wiener class given by

$$G(\omega) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{k\omega \mathrm{i}} \mathsf{F}_k, \quad \forall \omega \in \mathbb{R}.$$

From Lemma B.1 $G\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ is continuous, 2π-periodic, and $\{\mathsf{F}_k\}_{k\in\mathbb{Z}}$ is its sequence of Fourier coefficients. Consequently, $F = G$ because $F, G\colon \mathbb{R} \to \mathbb{C}^{M\times N}$ are two continuous 2π-periodic functions with the same sequence of Fourier coefficients. □

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