

Reducing dimensionality effect in importance sampling simulations

J.L. Sanz-González, S. Zazo and F. Álvarez-Vaquero

The dimensionality effect is avoided by the use of sufficient statistics in event probability estimators realised by importance sampling. If the system function is not a sufficient statistic, an approach is proposed to reduce the dimensionality effect in the estimators. Simulation results of false-alarm probability estimations, applied to radar detection, confirm a clear concordance with the theoretical results.

Introduction: As is well-known, the importance sampling (IS) technique [1–4] is a modified Monte-Carlo simulation applied to rare-event probability estimation, such as estimation of very low false-alarm probability in radar detection, or very low error-probability in communications.

Consider a random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ and its probability density function (PDF) $f_{\mathbf{X}}(\mathbf{x})$ defined in the R^n -space, being $\mathbf{x} = (x_1, x_2, \dots, x_n)$ a vector of real components. Define a statistic $t(\mathbf{x})$ and the critical region $t(\mathbf{x}) \geq t_0$ in the R^n -space; the probability $P_0 = \Pr\{t(\mathbf{X}) \geq t_0\} = E\{u(t(\mathbf{X}) - t_0)\}$, where $E\{\cdot\}$ means expectation with respect to $f_{\mathbf{X}}(\mathbf{x})$ and $u(\cdot)$ is the unit-step function. An alternative equation for this probability is $P_0 = E^*\{w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0)\}$, where $E^*\{\cdot\}$ means expectation with respect to $f_{\mathbf{X}}^*(\mathbf{x})$, known as the IS-PDF or the biasing PDF, and $w_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{X}}(\mathbf{x})/f_{\mathbf{X}}^*(\mathbf{x})$ is the weighting function. An estimator of P_0 is the sample mean \hat{P}_0 , i.e.

$$\hat{P}_0 = \frac{1}{N} \sum_{k=1}^N w_{\mathbf{X}}(\mathbf{x}_k^*) \cdot u(t(\mathbf{x}_k^*) - t_0) \quad (1)$$

being \mathbf{x}_k^* and $k=1, 2, \dots, N$, independent sample vectors with PDF $f_{\mathbf{X}}^*(\mathbf{x})$.

If $f_{\mathbf{X}}^*(\mathbf{x}) \neq 0$ wherever $f_{\mathbf{X}}(\mathbf{x}) \neq 0$ in $t(\mathbf{x}) \geq t_0$, $\mathbf{x} \in R^n$, the estimator \hat{P}_0 is unbiased and consistent [1]. In fact, the variance $\sigma_{\hat{P}_0}^2$ of \hat{P}_0 is

$$\sigma_{\hat{P}_0}^2 = E^*\{(\hat{P}_0 - P_0)^2\} = \frac{1}{N} \left(E^*\{[w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0)]^2\} - P_0^2 \right) \quad (2)$$

As is well-known [1], $f_{\mathbf{X}}^*(\mathbf{x}) = f_{\mathbf{X}}(\mathbf{x}) \cdot u(t(\mathbf{x}) - t_0)/P_0$ provides zero variance in (2) for any $N \geq 1$; however, it is not practical because P_0 is unknown (in fact, it has to be estimated). In the literature [1–4], some families of $f_{\mathbf{X}}^*(\mathbf{x})$ have been proposed for different estimation problems and the optimal solution is constrained to this family.

An important problem in many IS simulations is the dimensionality effect [2], meaning that $\sigma_{\hat{P}_0}^2$ increases as the dimension n of vector \mathbf{x} increases, for a constant number N of simulation runs.

Reducing the dimensionality effect: If P_0 and N are fixed in (2), to minimise $E^*\{[w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0)]^2\}$ is equivalent to minimising $\sigma_{\hat{P}_0}^2$.

Proposition 1. Define the weighting functions: $w_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{X}}(\mathbf{x})/f_{\mathbf{X}}^*(\mathbf{x})$ and $w_T(t) = f_T(t)/f_T^*(t)$, where $t = t(\mathbf{x})$ and the random variable $T = t(\mathbf{X})$ with $f_T(\cdot)$ and $f_T^*(\cdot)$ the corresponding induced PDF's; then, the following inequality holds:

$$E^*\{[w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0)]^2\} \geq E^*\{[w_T(T) \cdot u(T - t_0)]^2\} \quad (3)$$

Proof: Using conditional expectation, we can write

$$E^*\{[w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0)]^2\} = E^*\{E^*\{[w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0)]^2 | T\}\}$$

Considering that the variance is a non-negative number (or by Jensen's inequality), we can establish the following inequality:

$$E^*\{[w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0)]^2 | t\} \geq [E^*\{w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0) | t\}]^2$$

Also, $E^*\{w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0) | t\} = \int_{R^n} w_{\mathbf{X}}(\mathbf{x}) u(t(\mathbf{x}) - t_0) f_{\mathbf{X}}^*(\mathbf{x} | t) \mathbf{d}\mathbf{x}$ and $f_{\mathbf{X}}^*(\mathbf{x} | t) = f_{\mathbf{X}}^*(\mathbf{x}) f_T^*(t | \mathbf{x}) / f_T^*(t)$, $f_{T | \mathbf{X}}^*(t | \mathbf{x}) = f_{T | \mathbf{X}}(\mathbf{x} | t) = \delta(t - t(\mathbf{x}))$, $\delta(\cdot)$ being the Dirac delta function. Finally, after substituting the above, thus we have $E^*\{w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0) | t\} = w_T(t) \cdot u(t - t_0)$, and Proposition 1 has been proved.

Proposition 2: If $t(\mathbf{x})$ is a sufficient statistic for the family of $f_{\mathbf{X}}^*(\mathbf{x})$ and $f_{\mathbf{X}}(\mathbf{x})$, then

$$E^*\{[w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0)]^2\} = E^*\{[w_T(T) \cdot u(T - t_0)]^2\} \quad (4)$$

Proof: Owing to the property of sufficiency: $f_{\mathbf{X}}^*(\mathbf{x}) = g(\mathbf{x}) \cdot h^*(t(\mathbf{x}))$ and $f_{\mathbf{X}}(\mathbf{x}) = g(\mathbf{x}) \cdot h(t(\mathbf{x}))$, we have

$$E^*\{[w_{\mathbf{X}}(\mathbf{X}) \cdot u(t(\mathbf{X}) - t_0)]^2 | t\} = [h(t)/h^*(t) \cdot u(t - t_0)]^2 \quad (5)$$

From the PDF definition (assuming $dt > 0$ and $dt \rightarrow 0$)

$$f_T(t) dt = \Pr\{t \leq t(\mathbf{X}) < t + dt\} = \int_{t \leq t(\mathbf{x}) < t+dt} f_{\mathbf{X}}(\mathbf{x}) \mathbf{d}\mathbf{x} \\ = h(t) \int_{t \leq t(\mathbf{x}) < t+dt} g(\mathbf{x}) \mathbf{d}\mathbf{x}$$

and

$$f_T^*(t) dt = h^*(t) \int_{t \leq t(\mathbf{x}) < t+dt} g(\mathbf{x}) \mathbf{d}\mathbf{x}$$

we can write $w_T(t) = f_T(t)/f_T^*(t) = h(t)/h^*(t)$ and its substitution in (5) finishes the proof.

Proposition 1 means that the variance of \hat{P}_0 realised by $w_{\mathbf{X}}(\mathbf{x})$ in (1) is greater than the variance of \hat{P}_0 realised by $w_T(t)$ in place of $w_{\mathbf{X}}(\mathbf{x})$, unless $t(\mathbf{x})$ is a sufficient statistic according to Proposition 2, where the dimensionality effect is cancelled.

Finally, if $t(\mathbf{x})$ is not a sufficient statistic, we should compute or estimate the weighting function $w_T(t)$, in order to be used in place of $w_{\mathbf{X}}(\mathbf{x})$ in (1) for reducing the dimensionality effect in the estimation of P_0 . The computation of $w_T(t)$ should be performed through the computation of $f_T(\cdot)$ and $f_T^*(\cdot)$ over $t = t(\mathbf{x})$ from $f_{\mathbf{X}}(\mathbf{x})$ and $f_{\mathbf{X}}^*(\mathbf{x})$, respectively. This subject is now under research by the authors of this Letter.

Applications to radar detection: Consider the binary hypotheses H_0 (target absent) and H_1 (target present), defined as follows:

$$H_0: f_{\mathbf{X}}(\mathbf{x} | H_0) = \prod_{i=1}^n f_0(x_i) \quad \text{and} \quad H_1: f_{\mathbf{X}}(\mathbf{x} | H_1) = \prod_{i=1}^n f_1(x_i), \quad \text{where } \mathbf{x} = (x_1,$$

$x_2, \dots, x_n)$ is the vector of n independent envelope samples (n is the number of pulses per antenna beam width), $f_0(x)$ is the PDF of the noise sample and $f_1(x)$ is the PDF of the target signal with noise. For the Gaussian noise, $f_0(x)$ is the Rayleigh distribution; for a non-fluctuating (NF) target model, $f_1(x)$ is the Rice distribution with S as the signal-to-noise ratio parameter; for the Swerling II (SW-II) target model [5], $f_1(x)$ is the Rayleigh distribution with parameter S_0 the signal-to-noise ratio mean. The optimum Neyman-Pearson detector has the well-known structure [5]

$$t(\mathbf{x}) = \sum_{i=1}^n a(x_i) \begin{cases} > t_0 \\ < t_0 \end{cases} \quad \begin{matrix} H_1 \\ H_0 \end{matrix} \quad (6)$$

where $a(x) \simeq x$ (linear) for a NF target model, and $a(x) = x^2$ (quadratic) for the SW-II target model. The false-alarm probability (P_{fa}) is defined by $P_{fa} = \Pr\{t(\mathbf{X}) \geq t_0 | H_0\}$, hence, we can identify P_{fa} with P_0 and $f_{\mathbf{X}}(\mathbf{x} | H_0)$ with $f_{\mathbf{X}}(\mathbf{x})$ given above. According to Orsak [3], a good IS biasing PDF is $f_{\mathbf{X}}^*(\mathbf{x}) = f_{\mathbf{X}}(\mathbf{x} | H_1)$ with an adequate signal-to-noise ratio. Since the linear statistic is a sufficient statistic for detecting a NF target model, the adequate IS biasing PDF should correspond to the NF case, in order to cancel the dimensionality effect; on the other hand, quadratic statistic is a sufficient statistic for the SW-II target model, and the adequate IS biasing PDF should correspond to the SW-II case for cancelling the dimensionality effect.

Simulation results: Now, we consider the linear and quadratic statistics as detector statistics in (6), and estimate the false-alarm probability by using IS PDF's corresponding to the NF and SW-II target models. The results are shown in Figs. 1 and 2, where the parameter values are: number of integrated pulses, $n=8$; number of simulation runs, $N=1000$; IS-biasing parameters, $S=S_0=3$. In Fig. 1, we show P_{fa} -estimations against threshold t_0 for both detectors, using both biasing PDF's: NF (NF: dotted lines) and SW-II (SW-II: solid lines); in addition, we show theoretical curves (discontinuous red lines) of

P_{fa} -values (computed from integral equations). In Fig. 2, we show the corresponding relative-error estimation ($\hat{\epsilon}$) of P_{fa} -estimations:

$$\hat{\epsilon} = \sqrt{\hat{\sigma}_{\hat{P}_{fa}}^2} / \hat{P}_{fa}$$

for the four possibilities, where $\hat{\sigma}_{\hat{P}_{fa}}^2$ is a variance estimator of \hat{P}_{fa} . From Figs. 1 and 2, it can be seen that for the quadratic detector, \hat{P}_{fa} -values obtained by SW-II as IS-PDF are very close to the exact P_{fa} -values, and better than those obtained by the NF procedure. On the contrary, for the linear detector, the \hat{P}_{fa} provided by a NF as IS-PDF is better than that provided by the SW-II procedure. Similar results are obtained for $n = 16, 32$ or 64 .

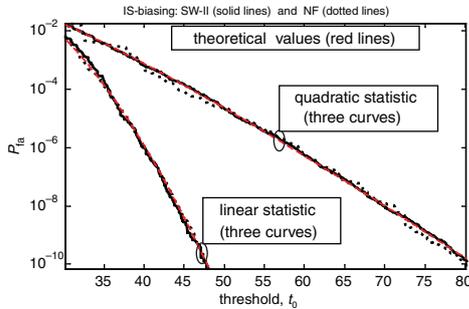


Fig. 1 False-alarm probability (P_{fa}) against threshold t_0 for linear and quadratic detectors: theoretical values and estimations by two different IS-PDF's (SW-II and NF)

Parameters: $n = 8, S = S_0 = 3$ and $N = 1000$

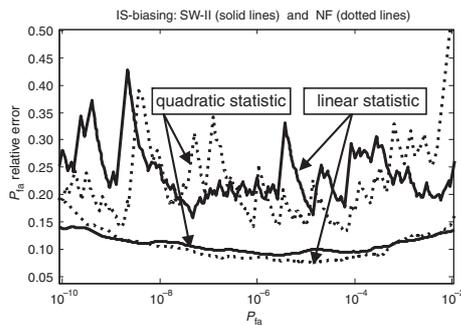


Fig. 2 P_{fa} -relative error ($\hat{\epsilon}$) against P_{fa} in same conditions of Fig. 1

Conclusion: It has been shown that if the system statistic is sufficient, the dimensionality effect is circumvented in rare-event probability estimation by the IS technique. Applications to false-alarm probability estimations in radar detectors have been given, and the simulation results are in agreement with the theoretical ones.

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One or more of the Figures in this Letter are available in colour online.

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